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О ВОССТАНОВЛЕНИИ ЗАШУМЛЕННЫХ СИГНАЛОВ МЕТОДОМ РЕГУЛЯРИЗАЦИИ

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Задача восстановления зашумленных сигналов рассматривается как задача вычисления значений неограниченного оператора. Применяется метод регуляризации Тихонова. Обсуждаются теоретические и практические подходы к проблеме выбора параметра регуляризации. Используется априорная информация о структуре искомого полезного сигнала. Материал излагается в терминах функционального анализа. Это позволяет обобщить полученные результаты на другие области обработки экспериментальных данных. Анализируется случай значительной зашумленности восстанавливаемых сигналов. Работа выполнена при поддержке РФФИ (проект 10–01–00297а).

Ключевые слова: регуляризация, пространство Гильберта, неограниченный оператор, слабая сходимость, сходимость по норме, неравенства Эйлера, тождества Эйлера.

Let L be a linear operator acting from H to G: $H \xrightarrow{L} G$. Here H and G are real Hilbert spaces. Let L be defined on a given nonempty convex set $D \subseteq H$. Suppose that the operator L is closed on D, i.e., for any sequence of elements $u_n \in D$: $u_n \xrightarrow{(H)} u_0$, $Lu_n \xrightarrow{(G)} g_0$, $n \to \infty$ (in other words, converging in H and G, respectively) we have $u_0 \in D$, $Lu_0 = g_0$. Let $u^* \in D$ be the sought "useful" signal given by its approximation $\tilde{u} \in H$: $u^* = \tilde{u} + \xi$, where $\xi \in H$ is an unknown "noise" whose size is specified in our further discussion.

One of the main mathematical problems in the reconstruction of noisy signals (let us call it problem R) can be formulated as follows: to construct an element $\hat{u} \in D$ such that the deviation

$$|u^* - \hat{u}| \equiv \left(\|Lu^* - L\hat{u}\|_G^2 + \|u^* - \hat{u}\|_H^2\right)^{1/2} \to 0$$

when the "noise" ξ also tends to zero (in a certain sense).

Note that the known element $\tilde{u}$ is not necessarily belong to an initially given set D defined as a sum of some known properties of the sought "useful" signal, such as, for example, the boundedness of the signal amplitude or its spectrum, nonnegativity, monotonicity, convexity, the absence of signals or frequencies on some intervals, the boundedness of the total variation, etc. These circumstances as well as the fact that the operator L can be unbounded (it is important, for example, if it is required to calculate some characteristics of a "useful" signal in the form of its derivative) allow us to conclude that the formulated problem is ill posed, i.e., the problem can be unsolvable in the classical sense and (or) can be unstable. It is possible to take, for example, $\hat{u} = P_D \tilde{u}$, where P_D is a projector in H onto the set D. If the set D is appropriately chosen, then it might happen that the above-defined element $\hat{u}$ is quite reasonable. However, our a priori knowledge on a "useful" signal u^*, as a rule, is rather rough and, therefore, the hope for such a favorable result is too optimistic. This is especially the case when the final result is strongly influenced by an unknown error in the registered signal. From the computational point of view, the determination of the projection onto the set D may be rather difficult.

1. In order to solve the above problem, we use the method of Tikhonov's regularization [1, 2]. In this connection, we define the following parametric functional:

$$\Phi_\alpha[u] \equiv \|u - \tilde{u}\|_H^2 + \alpha \|Lu - g\|_G^2, \quad u \in D.$$

Here $\alpha > 0$ is a regularization parameter and $g \in G$ is a given element. Let us consider the following auxiliary problem $\tilde{R}_\alpha$: to find an element $\tilde{u}_\alpha \in D$ such that $\inf\limits_{u \in D} \Phi_\alpha[u] = \Phi_\alpha[\tilde{u}_\alpha]$. According to the general theory of regularization [2, 3], we have

Theorem 1. *For any $\alpha > 0$, the auxiliary problem $\tilde{R}_\alpha$ is uniquely solvable and its solution satisfies the following variational Euler inequality:*

$$(\tilde{u}_\alpha - \tilde{u}, v - \tilde{u}_\alpha)_H + \alpha (L\tilde{u}_\alpha - g, L(v - \tilde{u}_\alpha))_G \geqslant 0 \quad \forall v \in D.$$

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Below problem $\tilde{R}_\alpha$ is called problem R_α if $\xi \equiv 0$. Let u_α be a solution to problem R_α. Then, obviously, the following variational Euler inequality similar to the previous one is valid:

$$(u_\alpha - u^*, w - u_\alpha)_H + \alpha(Lu_\alpha - g, L(w - u_\alpha))_G \geqslant 0 \quad \forall w \in D.$$

Assuming $v = u_\alpha$, $w = \tilde{u}_\alpha$ and adding both inequalities, after some simple manipulations we obtain the relation

$$\|\tilde{z}_\alpha\|_H^2 + \alpha\|L\tilde{z}_\alpha\|_G^2 \leqslant -(\tilde{z}_\alpha, \xi)_H, \tag{1}$$

where $\tilde{z}_\alpha \equiv \tilde{u}_\alpha - u_\alpha$. Similarly, the following estimate can be obtained from the second variational inequality for $w = u^*$:

$$\|z_\alpha\|_H^2 + \alpha\|Lz_\alpha\|_G^2 \leqslant -\alpha(Lu^* - g, Lz_\alpha)_H. \tag{2}$$

Here $z_\alpha \equiv u_\alpha - u^*$.

Theorem 2. *Let $\xi : \|\xi\|_H < \delta$. If the regularization parameter $\alpha = \alpha(\delta)$ is chosen in such a way that $\delta/\sqrt{\alpha} \to 0$, $\delta, \alpha \to 0$, then the limit relation $|\tilde{u}_\alpha - u^*| \to 0$ is valid as $\delta, \alpha \to 0$. In other words, algorithm $\tilde{R}_\alpha$ solves problem R.*

Proof. Indeed, according to (1) we have

$$\|\tilde{z}_\alpha\|_H^2 + \alpha\|L\tilde{z}_\alpha\|_G^2 \leqslant \delta\|\tilde{z}_\alpha\|_H;$$

hence,

$$\|\tilde{z}_\alpha\|_H \leqslant \delta, \quad \|L\tilde{z}_\alpha\|_G \leqslant \delta/\sqrt{\alpha}.$$

Then we get

$$|\tilde{u}_\alpha - u^*| \leqslant |z_\alpha| + |\tilde{z}_\alpha| \leqslant |z_\alpha| + \delta/\sqrt{\alpha}. \tag{3}$$

By virtue of the extremal property of the regularized solutions, we have

$$\|u_\alpha - u^*\|_H^2 + \alpha\|Lu_\alpha - g\|_G^2 \leqslant \|u^* - u^*\|_H^2 + \alpha\|Lu^* - g\|_G^2,$$

i.e., the following regularity conditions are satisfied [2]:

$$\|Lu_\alpha - g\|_G \leqslant \|Lu^* - g\|_G, \quad \|u_\alpha - u^*\|_H \to 0, \quad \alpha \to 0+\,.$$

In accordance with the general theory of regularization, the regularity conditions imply the limit relation

$$\lim_{\alpha\to 0+} |u_\alpha - u^*| = \lim_{\alpha\to 0+} |z_\alpha| = 0.$$

The proof of the theorem follows from this relation and from (3).

Remark 1. Theorem 2 is also valid when the regularization parameter α is chosen constructively in accordance with the residual principle [2]

$$\alpha : \|\tilde{u}_\alpha - \tilde{u}\|_H = \delta. \tag{4}$$

As is known [2], this scalar equation is solvable. There exist efficient numerical algorithms for searching a value of the regularization parameter according to the residual principle. The meaning of this principle is sufficiently obvious: if $\alpha \to 0$, then $\tilde{u}_\alpha \to P_D\tilde{u}$ and we might not obtain anything new in comparison with the known information; if $\alpha \to \infty$, then it is possible to prove that $\tilde{u}_\alpha$ tends to an element from the kernel of the operator L such that this element approximates $\tilde{u}$ in a mean-square sense. The realization of the residual principle results in a choice of the regularized solution consistent with the accuracy of measurements.

2. Theorem 2 does not ensure a guaranteed accuracy for the restoration of the signal u^*. Moreover, the convergence of $\tilde{u}_\alpha$ to u^* in norm $|\cdot|$ may be very slow.

Let the element v^* be such that the following “smoothness” variational condition is satisfied:

$$(Lu^* - g, L(u - u^*))_G \geqslant (h^*, u - u^*)_H \quad \forall u \in D, \quad h^* \in H. \tag{5}$$

Then the following theorem holds.

Theorem 3. *Let $\xi : \|\xi\|_H < \delta$. If the generalized “smoothness” condition* (5) *is fulfilled, then the following estimates are valid:*

$$\|z_\alpha\|_H \leqslant \alpha\|h^*\|_H, \quad \|Lz_\alpha\|_G \leqslant \sqrt{\alpha}\|h^*\|_H;$$

hence,

$$\min_{\alpha>0} |\tilde{u}_\alpha - u^*| \leqslant \text{const} \cdot \delta^{1/2}.$$

Proof. Indeed, assuming $u = u_\alpha$ in (5) and using (2), we have

$$\|z_\alpha\|_H^2 + \alpha\|Lz_\alpha\|_G^2 \leqslant -\alpha(h^*, z_\alpha)_H \leqslant \alpha\|h^*\|_H\|z_\alpha\|_H;$$

hence,

$$\|z_\alpha\|_H \leqslant \alpha\|h^*\|_H, \quad \|Lz_\alpha\|_G \leqslant \sqrt{\alpha}\|h^*\|_H.$$

According to (3), thus, we come to the inequality

$$|\tilde{u}_\alpha - u^*| \leqslant \text{const} \cdot \big(\sqrt{\alpha} + \delta/\sqrt{\alpha}\big).$$

From this we obtain the required estimate. The theorem is proven.

Remark 2. If the set D is linear, then inequality (5) becomes the identity

$$(Lu^* - g, Lv)_G = (h^*, v)_H \quad \forall v \in D.$$

If there exists a conjugate operator L^* (for example, D is a dense set in H) and the element $Lu^* - g \in D_{L^*}$ (D_{L^*} is the definition domain for L^*), then this identity yields $h^* = L^*(Lu^* - g)$.

Note that if $g \approx Lu^*$, then the accuracy of the restoration of an initial signal increases. This conclusion follows from the above estimates (in this case, $\|h^*\|_H$ is small).

In general, if $g = Lu^*$, u^* is the sought signal, then from the variational inequality for u_α it follows that $u_\alpha \equiv u^*$ $\forall \alpha > 0$, i.e., the regularized solutions are stationary. In the case of $g \approx Lu^*$, it is possible to expect that the range of “suitable” values of the regularization parameter is sufficiently wide. This significantly facilitates the reconstruction of a useful signal by the regularization method.

3. Now we assume that the set D is linear and dense in H. Instead of the variational Euler inequality for u_α, then, we have the Euler identity

$$(u_\alpha - u^*, w)_H + \alpha(Lu_\alpha - g, Lw)_G = 0 \quad \forall w \in D.$$

If $Lu^* - g \in D_{L^*}$, then, assuming $h^* = L^*(Lu^* - g)$, we obtain from this identity that

$$(z_\alpha + \alpha h^*, w)_H + \alpha(Lz_\alpha, Lw)_G = 0 \quad \forall w \in D,$$

i.e., the element z_α is the minimum of the functional

$$\alpha\|Lz\|_G^2 + \|z + \alpha h^*\|_H^2, \quad z \in D.$$

Theorem 4. *Let the set D be linear and dense in H and $\xi : \|\xi\|_H < \delta$. If $h^* = L^*(Lu^* - g) \in D_{L^*}$, then the following estimate holds:*

$$\min_{\alpha>0} |\tilde{u}_\alpha - u^*| \leqslant \text{const} \cdot \delta^{2/3}.$$

Proof. In accordance with the above remark, indeed, we take $z = -\alpha h^*$ as a comparison element. Then,

$$\alpha\|Lz_\alpha\|_G^2 + \|z_\alpha + \alpha h^*\|_H^2 \leqslant \alpha^3\|Lh^*\|_G^2.$$

Thus,

$$\|Lz_\alpha\|_G \leqslant \alpha\|Lh^*\|_G, \quad \|z_\alpha\|_H \leqslant \alpha\|h^*\|_H + \alpha^{3/2}\|Lh^*\|_G.$$

Since $|\tilde{z}_\alpha| \leqslant \delta/\sqrt{\alpha}$, $\alpha > 0$, we conclude that the theorem is proved for $\alpha = \delta^{2/3}$.

Remark 3. Large “smoothness” for the element u^* is required in Theorem 4. However, if we wish to obtain the regularized solutions with the guaranteed accuracy $\delta^{1/2}$ or $\delta^{2/3}$, then on a class of problems, from our point of view, the hypotheses of Theorems 3 and 4 cannot be relaxed [4]. It would be also desirable to eliminate the linearity of D , but preserving its convexity, and to formulate the smoothness conditions in a less burdensome variational form. For the regularized solutions, it is also important to obtain some estimates determined in accordance with the residual principle. Note that the signal u^* is reconstructed much more precisely than its L-characteristic.

4. Here we assume that D is a convex set in H and the "noise" $\xi \rightharpoonup 0$, i.e., the noise tends to zero in the sense of weak convergence in H. On the set D, we consider a linear invertible operator $\hat{L} : D \to G$ such that

$$c\|\hat{L}(u-v)\|_G \leqslant |u-v| \leqslant C\|\hat{L}(u-v)\|_G \quad \forall u,v \in D, \tag{6}$$

where the constants c and C do not depend on the choice of u, v and $0 < c \leqslant C < +\infty$. Then, from (1) it follows that

$$\|\tilde{z}_\alpha\|_H^2 + \alpha\|L\tilde{z}_\alpha\|_G^2 \leqslant -(\hat{L}^{-1}\hat{L}\tilde{z}_\alpha, \xi)_H = -(\hat{L}\tilde{z}_\alpha, (\hat{L}^{-1})^*\xi)_G \leqslant \frac{|\tilde{z}_\alpha|}{C}\,\Delta, \quad \Delta = \|(\hat{L}^{-1})^*\xi\|_G. \tag{7}$$

Therefore, obviously, the following theorem holds.

Theorem 5. *If $\Delta \to 0$ as $\xi \rightharpoonup 0$, then it is possible to specify a dependence $\alpha = \alpha(\Delta) \to 0$ such that*

$$|\tilde{u}_\alpha - u^*| \to 0, \quad \xi \rightharpoonup 0.$$

Remark 4. If the operator $(\hat{L}^{-1})^*$ is completely continuous, i.e., it maps any weak convergent sequence to a strongly convergent one, then, obviously, $\Delta \to 0$ as $\xi \rightharpoonup 0$. The latter, for example, is valid if any set of elements $u \in D_L$ is compact in H and $||u||_H \leqslant \text{const}$, $||Lu||_G \leqslant \text{const}$, where D_L is the definition domain of the operator L.

Thus, if $\xi \rightharpoonup 0$, then we may speak of the restoration of the signal u^* on the background of "large" noises. Indeed, if H is a separable space and e_n, $n = 1, 2, \ldots$, is its orthonormal basis, then $\xi_n = A_n e_n \rightharpoonup 0$ as $n \to \infty$ if $|A_n| \leqslant \text{const}$, i.e., this is the case of noises with large "amplitude" and high "frequency".

Remark 5. Some prior error estimates for the regularization in terms of Δ can be obtained if additional information on the "smoothness" of the useful signal u^* is available (see Theorems 3 and 4). This can be accomplished in a similar way. For brevity, we leave to the reader the derivation of the corresponding estimates.

We also note that inequalities (6) are satisfied for any L. In particular, if the operator L is boundedly invertible on D, then we may put $\hat{L} \equiv L$.

5. Let us consider problem $\hat{R}$: to find an element $\hat{u} \in D$, $\|\hat{u} - \tilde{u}\|_H \leqslant \delta$ such that it minimizes the functional $\Omega(u) \equiv \|Lu - g\|_G^2$, $u \in D$, i.e., $\inf\limits_u \Omega(u) = \Omega(\hat{u})$, where the element $u \in D$ is such that $\|u - \tilde{u}\|_H \leqslant \delta$.

Theorem 6. *If $\|\xi\|_H < \delta$, then a solution $\hat{u}$ to problem $\hat{R}$ exists and can be obtained by the regularization method.*

Proof. Indeed, let the regularization parameter α be chosen in accordance with the residual principle (4).

Then, for any element $u \in D$, $\|u - \tilde{u}\|_H \leqslant \delta$, we have (by virtue of the extremal property of regularized solutions):

$$\Phi_\alpha[\tilde{u}_\alpha] = \|\tilde{u}_\alpha - \tilde{u}\|_H^2 + \alpha\|L\tilde{u}_\alpha - g\|_G^2 = \delta^2 + \alpha\|L\tilde{u}_\alpha - g\|_G^2 \leqslant$$
$$\leqslant \|u - \tilde{u}\|_H^2 + \alpha\|Lu - g\|_G^2 \leqslant \delta^2 + \alpha\|Lu - g\|_G^2.$$

In other words,

$$\alpha\|L\tilde{u}_\alpha - g\|_G^2 \leqslant \alpha\|Lu - g\|_G^2 \quad \forall u \in D : \|u - \tilde{u}\|_H \leqslant \delta.$$

For $u = u^*$ in the previous inequalities, however, at the last stage we have the strict inequality (since $\|\xi\|_H < \delta$ by condition) and, consequently, $\alpha > 0$. Thus, the element $\tilde{u}_\alpha$ has the extremal property

$$\|L\tilde{u}_\alpha - g\|_G \leqslant \|Lu - g\|_G \quad \forall u \in D : \|u - \tilde{u}\|_H \leqslant \delta,$$

i.e., this element is a solution to problem $\hat{R}$. The theorem is proved.

Remark 6. Problem $\hat{R}$ has the following rather transparent pragmatic meaning: if a signal $\tilde{u}$ is given and it is required to find some of its characteristics (to determine the operator L), then it is natural to accomplish this with a minimum "price" $\Omega(u)$. The functional $\Omega(u)$ can also be considered as a "complexity" of the realization of a reconstructed signal. For example, for $L = d^2/dx^2$ we find a signal with a "minimum" curvature.

If it is necessary to preserve a "thin" structure of the useful signal, then its total variation [7] can be taken as $\Omega(u)$. Other useful approaches can also be applied by using, for example, a special set D [3, 5]. In general, an algorithm for the restoration of a useful signal may be complicated significantly. As a rule, the success corresponds to the situation when the operation P_D of projection onto the set D can be realized efficiently [6–8].

6. There exist efficient numerical methods [2, 3] to choose the regularization parameter in accordance with the residual principle (4). To accomplish this, however, it is necessary to know to sufficient accuracy a value of the parameter δ characterizing an error in the initial data $\tilde{u}$. In practice, unfortunately, this knowledge is not always available. Therefore, some heuristic (or pragmatic) approaches [10] can be used to choose the regularization parameter. For example, if there are two measurements $\tilde{u}_1$ and $\tilde{u}_2$ of a signal u^*, then it is possible to proceed as follows. We find the regularized solution $\tilde{u}^1_\alpha$ corresponding to $\tilde{u} = \tilde{u}_1$ and determine the regularization parameter in accordance with the following modified residual principle:

$$\|\tilde{u}^1_\alpha - \tilde{u}_2\| \to \min_{\alpha>0}.$$

If the measurements $\tilde{u}_1$ and $\tilde{u}_2$ are independent, then the modified residual principle yields rather satisfactory results. In its essence, this principle is close to the cross-validation method [10].

Another approach to determine the regularization parameter consists in the following. Let $u^{(1)}_\alpha$ be a solution to problem $\tilde{R}_\alpha$. The family $u^{(1)}_\alpha$ is said to be the initial regularizing family. Then, we put $g \equiv Lu^{(1)}_\alpha$ and solve problem $\tilde{R}_\alpha$ once more. Its solutions $u^{(2)}_\alpha$ is said to be the secondary regularizing family. If D is a linear set, then, obviously, the following Euler identities are valid:

$$(u^{(1)}_\alpha - \tilde{u}, v)_H + \alpha(Lu^{(1)}_\alpha - g, Lv)_G = 0,$$
$$(u^{(2)}_\alpha - \tilde{u}, v)_H + \alpha(Lu^{(2)}_\alpha - Lu^{(1)}_\alpha, Lv)_G = 0 \quad \forall v \in D.$$

From this we have

$$(u^{(2)}_\alpha - u^{(1)}_\alpha, v)_H + \alpha(L(u^{(2)}_\alpha - u^{(1)}_\alpha) - (Lu^{(1)}_\alpha - g), Lv)_G = 0 \quad \forall v \in D.$$

On the other hand, it is easy to prove the following identity:

$$\left(\frac{du^{(1)}_\alpha}{d\alpha}, v\right)_H + \alpha\left(L\,\frac{du^{(1)}_\alpha}{d\alpha}, Lv\right)_G + (Lu^{(1)}_\alpha - g, Lv)_G = 0 \quad \forall v \in D.$$

Multiplying the last identity by α and adding it to the previous one, we obtain

$$\left(u^{(2)}_\alpha - u^{(1)}_\alpha + \alpha\,\frac{du^{(1)}_\alpha}{d\alpha}, v\right)_H + \alpha\left(L(u^{(2)}_\alpha - u^{(1)}_\alpha) + \alpha L\,\frac{du^{(1)}_\alpha}{d\alpha}, Lv\right)_G = 0 \quad \forall v \in D.$$

Putting here

$$v = u^{(2)}_\alpha - u^{(1)}_\alpha + \alpha\,\frac{du^{(1)}_\alpha}{d\alpha},$$

we get

$$u^{(2)}_\alpha - u^{(1)}_\alpha + \alpha\,\frac{du^{(1)}_\alpha}{d\alpha} = 0, \quad L\left(u^{(2)}_\alpha - u^{(1)}_\alpha + \alpha\,\frac{du^{(1)}_\alpha}{d\alpha}\right) = 0. \tag{8}$$

Thus, the following the theorem holds.

Theorem 7. *If D is a linear set, then the secondary and initial regularizing families of solutions are related by* (8).

Note that some relations similar to (8) were obtained by the author more than 40 years ago. Based on these relations, a method for determining the regularization parameter (the so-called method of quasi-optimal values for the regularization parameter) was proposed in the 1960s by A. N. Tikhonov and V. B. Glasko [1]. Now we consider our interpretation of this method in terms of relations (8).

We call the modified method for determining the quasi-optimal values of the regularization parameter the method based on the following criteria:

$$\|u^{(2)}_\alpha - u^{(1)}_\alpha\|_H \to \min_{\alpha>0}, \quad \|L(u^{(2)}_\alpha - u^{(1)}_\alpha)\|_G \to \min_{\alpha>0},$$

or, more generally,

$$\rho\|u^{(2)}_\alpha - u^{(1)}_\alpha\|_H + (1-\rho)\|L(u^{(2)}_\alpha - u^{(1)}_\alpha)\|_G \to \min_{\alpha>0}.$$

Here $\rho : 0 \leqslant \rho \leqslant 1$ is an weighting factor. According to (8), the last criterion can be rewritten as

$$\rho\alpha\left\|\frac{du^{(1)}_\alpha}{d\alpha}\right\|_H + (1-\rho)\alpha\left\|L\,\frac{du^{(1)}_\alpha}{d\alpha}\right\|_G \to \min_{\alpha>0}. \tag{9}$$

Unfortunately, method (9) may yield several local extrema. Then, the chosen value of the regularization parameter has to be checked against the correspondence to the residual criterion. Note also that no knowledge on the secondary regularizing family of solutions is required in the realization of method (9).

Moreover, if we consider the so-called "geometrical" grid of values of the regularization parameter given as $\alpha_j = \tau\alpha_{j-1}$, where $j = 1, 2, \dots, M$, $\tau \approx 1$, $\tau < 1$, and $\alpha_0 > 0$ is a sufficiently large initial value, then we come to the following discrete variant:

$$\rho\|u^{(2)}_{\alpha_j} - u^{(1)}_{\alpha_{j-1}}\|_H + (1-\rho)\|L(u^{(2)}_{\alpha_j} - u^{(1)}_{\alpha_{j-1}})\|_G \to \min_{\alpha>0}. \tag{10}$$

Thus, the modified method for choosing the quasi-optimal values of the regularization parameter can be interpreted as a choice of the regularizing solution from a set of regularizing solutions in such a way that the chosen solution is most closely adjacent to other regularizing solutions. Note that this method is also valid for the case when D is a convex set.

7. The above regularization technique for solving the problem of restoration (filtration) of noisy signals is sufficiently general. For example, if the set $D = N_L = \{u \in D_L : Lu = 0\}$, then the regularization method can be reduced to the classical least-squares method.

If the information $\tilde{u}$ on a "useful" signal u^* is given in a discrete form (for example, as a set of values of some functionals), then we come to regularized polynomial or trigonometrical splines, depending on a choice of the operator L [9]. In the framework of our approach, we can also consider the exponential and Gaussian approximations, the wavelets and other approximation techniques. The application of Fourier transforms can appear to be especially efficient. The set D can be given as some conditions imposed on the spectral structure of the sought signals. Numerical results illustrating the above approaches to process digital signals are discussed in [11].

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