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Construction of the generalized iterative methods used for solution of the Fredholm integral equation

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Abstract: In this paper, we consider the Fredholm integral equations of the second kind and construct a new iterative scheme associated to the Nyström method, which was elaborated by Atkinson to approximate the solution over a large interval. Primarily, we demonstrate the inability to generalize the Atkinson iterative methods. Then, we describe our modified generalization in detail and discuss its advantages such as convergence of the iterative solution to the exact solution in the sense norm of the Banach space $C^0[a, b]$. Finally, we give a numerical examples to illustrate the accuracy and reliability of our generalization.

Keywords: Fredholm integral equations, numerical integration, iterative methods.

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Построение обобщенных итерационных методов, используемых для решения интегрального уравнения Фредгольма

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Аннотация: В данной работе мы рассматриваем интегральные уравнения Фредгольма второго рода и строим новую итерационную схему, связанную с методом Нистрема, который был разработан Аткинсоном для аппроксимации решения на большом интервале. Прежде всего, мы демонстрируем невозможность обобщения итерационных методов Аткинсона. Затем мы в деталях описываем наше модифицированное обобщение и обсуждаем его преимущества, такие как сходимость итерационного решения к точному в смысле нормы банахова пространства $C^0[a, b]$. Наконец, мы приводим численные примеры, иллюстрирующие точность и надёжность нашего обобщения.

Ключевые слова: интегральные уравнения Фредгольма, численное интегрирование, итерационные методы.

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1. Introduction. It is well known that integral equations are an inherent part of various fields of modern sciences and are an important part of pure mathematics, in particular [1]. For instance, they arise in many physical problems such as description of the radioactive transmission [2] and transport problems in astrophysics [3], the nuclear reactor theory [4, 5], the kinetic theory of gases [6], elasticity and fluid mechanics [7]. Recently this kind of integral equations has been applied to medical problems like the study of the development dynamics of Covide 19 [8]. Thus, there can be considered as a powerful mathematical tool for solving various modern scientific problems.

Today, with the rapid development of computer science, we find that many researchers have been able to devise a set of numerical methods that are best suited to this type of equations. A good example is the evaluation of the electric or magnetic field integral equation [9], which allows to obtain solutions close to exact ones. However, the main barrier remains because they require a large memory space to store the entire array in RAM. To solve this problem, scientists are developing iterative methods, which are easier to realize in comparison with direct methods. They require less memory space and less computing power and provide a good approximation of the solution as well.

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In this manuscript we are interested in the Fredholm integral equation of the second kind. This type of equations causes a great mathematical interest and is represented as follows: It is necessary to find a solution $u \in C^0[a,b]$ that satisfies the following equation

$$\forall\, t \in [a,b], \quad \lambda u(t) - \int_a^b k(t,s)u(s)ds = f(t), \tag{1}$$

where $\lambda \neq 0$ is a real or complex parameter, the kernel $k \in C^0([a,b]^2, \mathbb{R})$ and $f(t)$ is given function defined in the Banach space $C^0[a,b]$.

A great number of works is devoted to the search for the best possible numerical solution of the equation (1) by inventing new methods or by granulating and improving previously known methods. Let us mention here only a few of them:wavelet methods [2], Galerkin [10,11], collocation [12,13,14], quadrature [12], Chebyshev and Legendre collocation method [15], Rayleigh-Ritz method [16], deep learning [17], Ten-non polynomial cubic splines method [18], Gaussian process regression [19] and Taylor expansion [20].

The most famous existing method, which is considered one of the simplest, is the Nyström method [10, 12], where it proceeds by converting the equation (1) into an algebraic system whose size varies depending on the different divisions which we take on the interval $[a,b]$. The bigger size of the interval and the more divisions of $[a,b]$ which we take, the more time we consume to solve the algebraic system and the more space will need in the computer memory.

Despite the power and efficiency of the previous methods, they have some problems, including the consumption of a large space in the computer memory and sometimes the consumption of time in the resolution. To solve this problem, the iterative methods were proposed. The first to suggest this approach was Atkinson. Since then, the idea of iterative methods has been firmly established in the generally accepted methodology, see e.g. [21].

Atkinson in his article [22] suggested two iterative methods. In this manuscript, we follow principles proposed by him and combine these two methods to find the best version of the possible iterative scheme. We review some theorems that prove the convergence of the solution. We end our article with examples that illustrate the importance of the circular we have made and the effectiveness of the studied approach as well.

2. Atkinson Method. Before we start, let us focus here on some important points that will be used by us hereafter. First of all, we define the norm of the Banach space $C^0[a,b]$ in the following way:

$$\forall v \in C^0[a,b], \quad \|v\|_{C^0[a,b]} = \max_{a \leqslant t \leqslant b} |v(t)|.$$

We denote by $BL(C^0[a,b])$ the Banach space of linear and bounded operators, which defined in $C^0[a,b]$ in itself. It is equipped with the following norm:

$$\forall A \in BL(C^0[a,b]), \quad \|A\| = \sup_{\|v\|_{C^0[a,b]} \leqslant 1} \|Av\|_{C^0[a,b]}.$$

Now, we define the linear operator K as

$$K: \ C^0[a,b] \longrightarrow C^0[a,b], \quad v \longrightarrow Kv(t) = \int_a^b k(t,s)v(s)\, ds. \tag{2}$$

Then, the equation (12) may be rewritten in the next equivalent form:

$$(\lambda I - K)u = f, \tag{3}$$

where I is the identity operator of the Banach space $C^0[a,b]$.

If the condition $|\lambda| > \|K\|$ is fulfilled, then it follows from the Neumann's theorem [10] that $(\lambda I - K)^{-1}$ exists and is bounded. Thus, equation (3) has a unique solution in $C^0[a,b]$. This means that the existence and uniqueness of the analytical solution is ensured. This allows us to go on to the construction of our generalized method and the search for our iterative solution, which satisfies the integral equation (3). For all $n \geqslant 1$, Δ_n is the uniform discretization of the interval $[a,b]$:

$$\forall\, n \geqslant 1, \quad \Delta_n = \Big\{ a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b, \quad h = t_{j+1} - t_j, \quad 0 \leqslant j \leqslant n \Big\}.$$

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As the first numerical solution of (3), we propose the Nyström solution $\{u_n\}_{n\geqslant 1}$, which obeys the following approximate equation:

$$\lambda u_n = K_n u_n + f. \tag{4}$$

Here K_n is the numerical integral operator or Nyström operator given as:

$$\forall n \geqslant 1, \quad \forall v \in C^0[a,b], \quad \forall t \in [a,b], \quad K_n v(t) = \sum_{j=0}^{n} \omega_j k(t,t_j) v(t_j), \tag{5}$$

where $\{\omega_j\}_{j=0}^n$ are called weights, be selected depending on the used numerical scheme. For example, Gaussian numerical scheme or the trapezoidal rule, or Simpson's rule [12]. But, for all choices $\{\omega_j\}_{j=0}^n$ the following condition should be satisfied:

$$\exists W > 0, \quad \sup_{n\geqslant 1} \sum_{j=0}^{n} |\omega_j| = W < +\infty.$$

As was mentioned earlier, the larger the number n, the more time we will spend on the solution and the more memory space we will consume. To solve this problem, we offer the first iterative method proposed by Atkinson [22], which is based on a choice of two division numbers of the interval $[a,b]$, namely n and m, such that $m \gg n$. We use the iterative process to calculate the solution u_m. Thus, for all $\nu \geqslant 1$ the first iterative solution $\{u_m^\nu\}_{m\gg n}$ meets the following equation:

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n) u_m^{\nu+1} = (K_m - K_n) u_m^\nu + f, \quad \nu \geqslant 1. \end{cases} \tag{6}$$

The idea of the second Atkinson iterative method [22] comes through the substitution of u_m, which is satisfied by the equation:

$$u_m = \frac{1}{\lambda} K_m u_m + \frac{1}{\lambda} f,$$

in the right side of the formula:

$$(\lambda I - K_n) u_m = (K_m - K_n) u_m + f. \tag{7}$$

Then, the second iterative scheme of Atkinson reads

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n) u_m^{\nu+1} = \dfrac{1}{\lambda}(K_m - K_n) K_m u_m^\nu + \dfrac{1}{\lambda}(K_m - K_n) f + f, \quad \forall \nu \geqslant 1. \end{cases} \tag{8}$$

Two iterative methods of Atkinson are more simple because they are limited. We can't go further than the second method. Our goal is to increase the value of $(K_m - K_n)u_m$ to avoid loss of precision and poor convergence to zero. For this reason, we apply the same steps of Atkinson. It follows from equation (9) that

$$u_m = \frac{1}{\lambda}(K_m - K_n) u_m + \frac{1}{\lambda} K_n u_m + \frac{1}{\lambda} f. \tag{9}$$

We substitute it in the left side of equation (7) and get the third iterative scheme $\forall\, m \gg n$:

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n) u_m^{\nu+1} = \dfrac{1}{\lambda}(K_m - K_n)^2 u_m^\nu + \dfrac{1}{\lambda}(K_m - K_n) K_n u_m^\nu + f + \dfrac{1}{\lambda}(K_m - K_n) f, \quad \forall \nu \geqslant 1. \end{cases} \tag{10}$$

In a similar way, substitution of the equation (9) in the right side of the system (10) leads us to the fourth iterative scheme $\forall\, m \gg n$, presented in the following form:

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n) u_m^{\nu+1} = \dfrac{1}{\lambda^2}(K_m - K_n)^3 u_m^\nu + \dfrac{1}{\lambda^2}(K_m - K_n)^2 K_n u_m^\nu + \dfrac{1}{\lambda}(K_m - K_n) K_n u_m^\nu \\ \qquad\qquad + f + \dfrac{1}{\lambda}(K_m - K_n) f + \dfrac{1}{\lambda^2}(K_m - K_n)^2 f, \qquad \forall \nu \geqslant 1. \end{cases} \tag{11}$$

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If we repeat the same process p times, we will obtain the first generalization scheme for all $p \geqslant 1$:

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n)u_m^{\nu+1} = \dfrac{1}{\lambda^{p-1}}(K_m - K_n)^p u_m^\nu + \displaystyle\sum_{q=1}^{p-1} \frac{1}{\lambda^q}(K_m - K_n)^q K_n u_m^\nu \\ \qquad\qquad + \displaystyle\sum_{q=1}^{p-1} \frac{1}{\lambda^q}(K_m - K_n)^q f + f, \quad \forall \nu \geqslant 1. \end{cases} \tag{12}$$

At $p = 1$ we get the first Atkinson's iterative scheme (8) and at $p = 2$ the third one (10). We try to find a method combining three Atkinson schemes. Despite their advantages, there remains the problem of the existence of $(K_m - K_n)K_n$. Indeed, in Atkinson's paper [21], the convergence of the iterative solution depends on the convergence of $(K_m - K_n)K_n$. Moreover, if one takes a value of p strictly greater than 2, one observes the divergence of the error from zero. As a result, the best solution can be obtained at $p = 2$, which is the second Atkinson scheme.

3. Generalization Procedures. In this section, we review the basic steps that allow us to build the appropriate iterative scheme. We also present some theorems that prove the convergence of the proposed iterative scheme.

Using (9), we can rewrite u_m in the following way:

$$u_m = (\lambda I - K_n)^{-1}(K_m - K_n)u_m + (\lambda I - K_n)^{-1} f. \tag{13}$$

Substituting (13) in the left side of (9), we find:

$$\begin{aligned} (\lambda I - K_n)u_m &= (K_m - K_n)\Big((\lambda I - K_n)^{-1}(K_m - K_n)u_m + (\lambda I - K_n)^{-1} f\Big) + f, \\ &= (K_m - K_n)(\lambda I - K_n)^{-1}(K_m - K_n)u_m + (K_m - K_n)(\lambda I - K_n)^{-1} f + f. \end{aligned} \tag{14}$$

Now, we substitute the formula of u_m given by (1) in the left side of the last equation (2), we get the following equation

$$\begin{aligned} (\lambda I - K_n)u_m &= (K_m - K_n)(\lambda I - K_n)^{-1}(K_m - K_n)\Big[(\lambda I - K_n)^{-1}(K_m - K_n)u_m + (\lambda I - K_n)^{-1} f\Big] \\ &+ (K_m - K_n)(\lambda I - K_n)^{-1} f + f \\ &= \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^2 (K_m - K_n)u_m + \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^2 f \\ &+ (K_m - K_n)(\lambda I - K_n)^{-1} f + f. \end{aligned} \tag{15}$$

We continue to repeat the same process p times. So that, for each new equation we obtain, we substitute in the left part the value of u_m given by (1) and get:

$$(\lambda I - K_n)u_m = \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^p (K_m - K_n)u_m + \sum_{q=1}^{p} \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^q f + f. \tag{16}$$

Finally, $\forall p \geqslant 0$ and $\forall \nu \geqslant 1$ we can present our generalization iterative system in the following form:

$$\begin{cases} u_n^0 \in C^0[a,b], \\ (\lambda I - K_n)u_m^{\nu+1} = \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^p (K_m - K_n)u_m^\nu + \sum\limits_{q=1}^{p} \Big[(K_m - K_n)(\lambda I - K_n)^{-1}\Big]^q f + f. \end{cases} \tag{17}$$

Now, using the last scheme, our goal will be to search for the best value of p that can give us the best numerical solution. We specify that the increase in the value of p is not related to the decrease of the error to zero. For instance, one can obtain the best numerical solution at $p = 5$ without further increasing the value of power p. This is explained in the examples below.

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4. Convergence Analysis. Now, we have to prove that the iterative solution approaches the exact solution. For this purpose, in this section we consider theorems proving the convergence of the iterative scheme (17). Primarily, we show the following convergence

$$u_m^\nu \to u_m, \quad \text{when} \quad \nu \to +\infty. \tag{18}$$

Then, we prove that

$$u_m^\nu \to u, \quad \text{when} \quad \nu \to +\infty \quad \text{and} \quad m \to +\infty. \tag{19}$$

To demonstrate convergence of the Nyström method, we need to prove the existence of $(\lambda I - K_n)^{-1}$. To do this, one first has to demonstrate that the sequence $\{K_n\}_{n\geqslant 1}$ is ν-convergent to K. In this regard, let us turn to the following theorem:

Theorem 1 *Let K be a compact operator defined by (5) and let $\{K_n\}_{n\geqslant 1}$ be an approximation operator converging pointwise to K. Then,*

$$\lim_{n\to\infty} \| (K - K_n)K \| = 0. \tag{20}$$

Proof 1 Since K is a compact one, then $S = \left\{Kv, \|v\|_{C^0[a,b]} \leqslant 1\right\}$ has a compact closure in $C^0[a,b]$. Using definition of the operator norm, we obtain

$$\| (K - K_n)K \| = \sup_{\|v\|_{C^0[a,b]\leqslant 1}} \|(K - K_n)Kv\|_{C^0[a,b]} = \sup_{y\in S} \|(K - K_n)y\|_{C^0[a,b]}.$$

Then, it follows from the Banach-Steinhaus theorem [10] that K_n converges uniformly to K on S.

Now, we consider the next lemma to prove that $(\lambda I - K_n)^{-1}$ exists and is bounded.

Lemma 1 *For n large enough, let $\{K_n\}_{n\geqslant 1}$ be the Nyström approximation operator converging pointwise to K and*

$$\lim_{n\to\infty} \| (K - K_n)K \| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \| (K - K_n)K_n \| = 0. \tag{21}$$

Then $(\lambda I - K_n)^{-1}$ exists and is bounded.

Proof 2 See Atkinson [12].

In the rest of our convergence analysis, we need to show the convergence of $[(K_m - K_n)(\lambda I - K_n)^{-1}]^p$, $\forall\, p \geqslant 0$ to zero. For this reason we give the following theorem

Theorem 2 *Let $\{K_m\}_{m\geqslant 1}$ and $\{K_n\}_{n\geqslant 1}$ be two Nyström approximation operators and let K be an integral operator defined by (5), then*

$$\forall\, p \geqslant 0, \quad [(K_m - K_n)(\lambda I - K_n)^{-1}]^p \to 0, \quad \textit{when} \quad m, n \to +\infty. \tag{22}$$

Proof 3 For all $n \geqslant 1$, we have that $(\lambda I - K_n)^{-1}$ is bounded, so $\exists\, C > 0$ such that

$$\|(\lambda I - K_n)^{-1}\| \leqslant C.$$

Therefore,

$$\|[(\lambda I - K_n)^{-1}]^p\| \leqslant \|(\lambda I - K_n)^{-1}\|^p \leqslant C^p,$$

which is proved that $[(\lambda I - K_n)^{-1}]^p$ is bounded.

We set $A = [(\lambda I - K_n)^{-1}]^p$. For all $n, m \geqslant 1$ and $n \ll m$ the following equality is valid:

$$[(K_m - K_n)(\lambda I - K_n)^{-1}]^p = (K_m - K_n)^p A = (K_m - K_n)(K_m - K_n)\ldots(K_m - K_n)A.$$

On the other hand, we have two sequences operators $\{K_n\}_{n\geqslant 1}$ and $\{K_m\}_{m\geqslant 1}$ of finite rank, this is implies that they are compact. Then, we can say that $(K_m - K_n)$ is compact. So, $(K_m - K_n)A$ is compact, which gives that it is bounded. Now, since $(K_m - K_n)A$ is bounded and $(K_m - K_n)$ is compact, we obtain that $(K_m - K_n)(K_m - K_n)A$ is compact. Then one can conclude that $(K_m - K_n)^{p-1}A$ is compact.

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We put $B = (K_m - K_n)^{p-1}A$. Let define the set P as

$$P = \Big\{ Bv,\ \|v\|_{C^0[a,b]} \leqslant 1 \Big\}.$$

Since B is compact, the set P has a compact closure in $C^0[a,b]$. The definition of the operator norm yields

$$\|(K_m - K_n)B\| \quad \leqslant \quad \|(K_m - K)B\| + \|(K - K_n)B\|.$$

From theorem 1 follows that $\{K_m\}_{m\geqslant 1}$ and $\{K_n\}_{n\geqslant 1}$ converge uniformly to K in P. Therefore,

$$\lim_{m,n\to+\infty} \|(K_m - K_n)B\| = 0.$$

In the next theorem we show that the iterative solution of system (17) converges to u_m.

Theorem 3 *Let n and m large enough, such that $m \gg n$, $\nu \geqslant 1$ and u_m^ν be an iterative solution of (17) and u_m be a solution of (16). Then*

$$\|u_m^{\nu+1} - u_m\|_{C^0[a,b]} \leqslant \left[\|(\lambda I - K_n)^{-1}\| \left\| \left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^p (K_m - K_n) \right\| \right]^{\nu+1} \|u_m^0 - u_m\|_{C^0[a,b]}. \quad (23)$$

Proof 4 Let m and n large enough, such that $m \gg n$ and $\nu \geqslant 1$, we have

$$\begin{aligned}(\lambda I - K_n)(u_m^{\nu+1} - u_m) = &\left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^p (K_m - K_n)u_m^\nu + \sum_{q=1}^{p}\left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^q f + f \\ &- \left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^p (K_m - K_n)u_m - \sum_{q=1}^{p}\left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^q f - f.\end{aligned}$$

Which is equivalent to

$$u_m^{\nu+1} - u_m = (\lambda I - K_n)^{-1}\left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^p (K_m - K_n)(u_m^\nu - u_m). \quad (24)$$

Therefore,

$$\| u_m^{\nu+1} - u_m \|_{C^0[a,b]} \leqslant \| (\lambda I - K_n)^{-1} \| \| [(K_m - K_n)(\lambda I - K_n)^{-1}]^p (K_m - K_n) \| \| u_m^\nu - u_m \|_{C^0[a,b]}. \quad (25)$$

Using the analogous procedure, one can get

$$u_m^\nu - u_m = (\lambda I - K_n)^{-1}\left[(K_m - K_n)(\lambda I - K_n)^{-1}\right]^p (K_m - K_n)(u_m^{\nu-1} - u_m). \quad (26)$$

Then,

$$\| u_m^\nu - u_m \|_{C^0[a,b]} \leqslant \| (\lambda I - K_n)^{-1} \| \| [(K_m - K_n)(\lambda I - K_n)^{-1}]^p (K_m - K_n) \| \| u_m^{\nu-1} - u_m \|_{C^0[a,b]}. \quad (27)$$

Taking into account (27) and (25), we obtain

$$\| u_m^{\nu+1} - u_m \|_{C^0[a,b]} \leqslant \Big[\| (\lambda I - K_n)^{-1} \| \| [(K_m - K_n)(\lambda I - K_n)^{-1}]^p (K_m - K_n) \| \Big]^2 \| u_m^{\nu-1} - u_m \|_{C^0[a,b]}. \quad (28)$$

By recurrence, one can prove the following inequality:

$$\| u_m^{\nu+1} - u_m \|_{C^0[a,b]} \leqslant \Big[\| (\lambda I - K_n)^{-1} \| \| [(K_m - K_n)(\lambda I - K_n)^{-1}]^p (K_m - K_n) \| \Big]^{\nu+1} \| u_m^0 - u_m \|_{C^0[a,b]},$$

where u_m^0 is the initial solution given by Nyström method.

Since

$$\forall \varepsilon > 0,\ \exists\, n_0 \geqslant n:\ \| (\lambda I - K_n)^{-1} \| \| [(K_m - K_n)(\lambda I - K_n)^{-1}]^p (K_m - K_n) \| \leqslant \varepsilon\ < 1,$$

then

$$\| u_m^{\nu+1} - u_m \|_{C^0[a,b]} \to 0, \quad \text{when} \quad \nu \to +\infty.$$

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Finally, the next corollary shows that the iterative solution $u_m^{\nu+1}$ of the system (17) converges to an exact solution u of equation (12).

Corollary 1 For all $m \geqslant 1$ and $\nu \geqslant 1$ let $u_m^{\nu+1}$ be an iterative solution of (17) and let u be an exact solution of (12), then

$$\lim_{m\to+\infty}\left(\lim_{\nu\to+\infty}\|u_m^{\nu+1}-u\|_{C^0[a,b]}\right)=0. \tag{29}$$

Proof 5 For all m large enough and $\nu \geqslant 1$, we have

$$\|u_m^{\nu+1}-u\|_{C^0[a,b]} \leqslant \|u_m^{\nu+1}-u_m\|_{C^0[a,b]}+\|u_m-u\|_{C^0[a,b]}.$$

Thus, using theorem 3 and the convergence of u_m to u, which is clearly proved in [12], we obtain the result.

5. System Performance. To program our generalization represented by (17), we need to reformulate it to a new form. For this aim, we introduce the residue r

$$r^\nu=\lambda u_m^\nu-K_m u_m^\nu-f,\quad \nu\geqslant 1$$

and write the following chain of equalities:

$$\begin{aligned}
(\lambda I-K_n)u_m^{\nu+1} &= [(K_n-K_m)(\lambda I-K_n)^{-1}]^p[(K_n-K_m)u_m^\nu+f]+\sum_{q=1}^{p-1}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f\\
&= [(K_n-K_m)(\lambda I-K_n)^{-1}]^p[r^\nu+(\lambda I-K_n)u_m^\nu]+\sum_{q=1}^{p-1}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f\\
&= [(K_n-K_m)(\lambda I-K_n)^{-1}]^{p-1}[(K_n-K_m)(\lambda I-K_n)^{-1}u_m^\nu+(K_n-K_m)u_m^\nu]\\
&+ \sum_{q=1}^{p-1}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f\\
&= [(K_n-K_m)(\lambda I-K_n)^{-1}]^{p-1}[(K_n-K_m)(\lambda I-K_n)^{-1}r^\nu+(K_n-K_m)u_m^k+f]\\
&+ \sum_{q=1}^{p-2}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f\\
&= [(K_n-K_m)(\lambda I_n-K_n)^{-1}]^{p-1}[r^\nu+(K_n-K_m)(\lambda I-K_n)^{-1}u_m^\nu+(\lambda I-K_n)u_m^\nu]\\
&+ \sum_{q=1}^{p-2}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f\\
&= [(K_n-K_m)(\lambda I_n-K_n)^{-1}]^{p-2}\Big[(K_n-K_m)(\lambda I-K_n)^{-1}r^\nu\\
&+ [(K_n-K_m)(\lambda I-K_n)^{-1}]^2+(K_n-K_m)u_m^\nu\Big]+\sum_{q=1}^{p-2}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q f+f.
\end{aligned}$$

This is equivalent to

$$(\lambda I-K_n)u_m^{\nu+1}=(\lambda I-K_n)u_m^\nu+r^\nu+\sum_{q=1}^{p}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q r^\nu,\quad \nu\geqslant 1.$$

We set

$$\delta=(\lambda I-K_n)^{-1}\sum_{q=1}^{p}\left[(K_n-K_m)(\lambda I-K_n)^{-1}\right]^q r^\nu.$$

Finally, we get a new system equivalent to system (4), which has the following form for all $\nu\geqslant 1$

$$\left\{\begin{aligned}
r^\nu &= \lambda u_m^\nu-K_m u_m^\nu-f,\\
\delta^\nu &= (\lambda I-K_n)^{-1}\sum_{q=1}^{p}[(K_n-K_m)(\lambda I-K_n)^{-1}]^q r^\nu,\\
u_m^{\nu+1} &= u_m^\nu+(\lambda I-K_n)^{-1}r^\nu+\delta^\nu.
\end{aligned}\right. \tag{30}$$

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In the next part we will demonstrate the principle by which the solution of the system (30) is found. Let U_m^ν be a discretized approximation of u satisfying the system (30). Now determine the following vectors

$$\begin{aligned}
U_m^\nu &= (u_m^\nu(t_0), u_m^\nu(t_1), \ldots, u_m^\nu(t_m)),\\
U_n^\nu &= (u_n^\nu(t_0), u_n^\nu(t_1), \ldots, u_n^\nu(t_n)),\\
F &= (f(t_0), f(t_1), \ldots, f(t_m)),\\
\delta^\nu &= (\delta^\nu(t_0), \delta^\nu(t_1), \ldots, \delta^\nu(t_m)),\\
r_m^\nu &= (r^\nu(t_0), r^\nu(t_1), \ldots, r^\nu(t_m))
\end{aligned}$$

and matrices

$$\begin{aligned}
A_{n\times n} &= w_j K(t_i, t_j), \quad 0 \leqslant i \leqslant n, \quad 0 \leqslant j \leqslant n,\\
B_{m\times m} &= w_j K(t_i, t_j), \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant m,\\
C_{n\times m} &= w_j K(t_i, t_j), \quad 0 \leqslant i \leqslant n, \quad 0 \leqslant j \leqslant m,\\
D_{m\times n} &= w_j K(t_i, t_j), \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant n,\\
I_{n\times n} &= \text{the identity matrix of size } n.
\end{aligned}$$

Iteration 0:

It is necessary to solve the following linear system

$$(\lambda I_{n\times n} - A_{n\times n}) U_n^0 = F.$$

We calculate

$$\begin{aligned}
\bar{U}_n(i) &= \frac{1}{\lambda}\left[\sum_{j=0}^{n} w_j K(t_i, t_j) U_n(j) + F(i)\right], \quad 0 \leqslant i \leqslant m,\\
U_m^0(i) &= \frac{1}{\lambda}\left[\sum_{j=0}^{m} w_j K(t_i, t_j) \bar{U}_n(j)\right], \quad 0 \leqslant i \leqslant m
\end{aligned}$$

and find a residue

$$r_m^0(i) = \lambda U_m(i) - \sum_{j=0}^{m} w_j K(t_i, t_j) U_m(j) - F(i), \quad 0 \leqslant i \leqslant m \ \text{ or } \ 0 \leqslant i \leqslant n.$$

Iteration $\nu \geqslant 1$:

While err > tolerance

1. We calculate δ^ν. To do this, we follow the next steps:

   (a) δ^ν is determined as

   $$\delta^k = (\lambda I_{n\times n} - A_{n\times n})^{-1} \sum_{q=1}^{p} \left[(A_{n\times n} - C_{n\times m})(\lambda I_{n\times n} - A_{n\times n})^{-1} (r_n^\nu)^{\frac{1}{p}}\right]^q.$$

   (b) We put $z = (\lambda I_{n\times n} - A_{n\times n})^{-1} (r_n^\nu)^{\frac{1}{p}}$ and solve the following algebraic system

   $$(\lambda I_{n\times n} - A_{n\times n}) z = (r_n^\nu)^{\frac{1}{p}}.$$

   (c) Then we compute

   $$z(t_i) = \frac{1}{\lambda}\left[\sum_{j=0}^{n} w_j K(t_i, t_j) z(t_j) + y(t_i)\right], \quad 0 \leqslant i \leqslant m \text{ or } 0 \leqslant i \leqslant m.$$

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(d) After that we find the following vector M:

$$M_{n\times 1} = \sum_{q=1}^{p}\Big[(A_{n\times n} - C_{n\times m})z\Big]^q,$$

$$M_{m\times 1} = \sum_{q=1}^{p}\Big[(D_{m\times n} - B_{m\times m})z\Big]^q,$$

(e) Solve the next system

$$(\lambda I_{n\times n} - A_{n\times n})\delta^k = M_{n\times 1},$$

$$\delta^\nu(i) = \frac{1}{\lambda}\left[\sum_{j=0}^{n} w_j K(t_i,t_j)\delta^\nu(j) + M_{m\times 1}(i)\right], \quad 0 \leqslant i \leqslant m.$$

2. We set $G = (\lambda I_{n\times n} - A_{n\times n})^{-1} r^\nu$ and solve the system

$$(\lambda I_{n\times n} - A_{n\times n})G = r^\nu,$$

$$G(i) = \frac{1}{\lambda}\left[\sum_{j=0}^{n} w_j K(t_i,t_j)G(j) + r_m^\nu(i)\right], \quad 0 \leqslant i \leqslant m.$$

3. The solution of the $\nu+1$ order is given as

$$U_m^{k+1} = U_m^\nu + G + \delta^\nu,$$

4. Finally, we compute the error

$$err = \max_{0\leqslant i\leqslant m} |U_m^{\nu+1}(i) - U_m^\nu(i)|.$$

Return

6. Numerical Tests. To illustrate the efficiency and accuracy of our proposed method, we provide three examples below. The corresponding errors are defined as

$$err^\nu = \max_{0\leqslant i\leqslant n} |U_m^{\nu+1}(i) - U_m^\nu(i)| \tag{31}$$

and are shown in tables below.

Now we consider the following equation:

$$\forall t \in [0,b], \quad \lambda u(t) = \int_0^b \frac{u(s)}{(s^3+t)^2+1}\, ds + f(t), \tag{32}$$

where $f(t) = 2t^2 - \dfrac{1}{3}\arctan(b^3+t) + \dfrac{1}{3}\arctan(t)$ and $\lambda = 2$. In table 1 we present the error values of equation (32) for different values of b, m, ν and p.

Let us turn to another example

$$\forall t \in [0,b], \quad \lambda u(t) = \int_0^b \frac{u(s)}{e^t+e^s}\, ds + f(t), \tag{33}$$

where $f(t) = e^t + \log\left(\dfrac{1+e^t}{e^b+e^t}\right)$ and $\lambda = 1$. In table 2 we give the error values of equation (33) for different values of b, m, ν and p.

Finally, we consider the third equation

$$\forall t \in [-a,a], \quad \lambda u(t) = \int_{-a}^{a} \frac{u(s)}{200+\cos(s)+t}\, ds + f(t), \tag{34}$$

where $f(t) = \sin(t)$ and $\lambda = 1$. In table 3 we present the error values of equation (34) for different values of b, m, ν and p.

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Table 1. The error between the exact and approximate solution of equation (32) for different values of m, ν and p

b	m	p	ν			
			10	20	30	40
100	600	1	3.7380e−10	1.0176e−15	1.0176e−15	1.0176e−15
		2	2.3882e−11	1.7369e−17	2.2493e−25	2.4989e−32
		3	5.3737e−10	2.5513e−17	4.1540e−25	1.1410e−31
		4	4.3764e−11	1.3879e−17	6.3256e−26	1.3646e−32
		5	4.6196e−11	5.5512e−17	1.8789e−24	4.3605e−31
		6	4.5546e−11	5.5513e−17	3.454e−25	7.8779e−32
		7	4.5716e−11	5.5513e−17	3.4569e−25	7.9362e−32
		8	4.5617e−11	4.5505e−17	6.3220e−27	1.4929e−33
		9	4.5683e−11	5.5512e−17	5.0637e−25	1.1616e−31
		10	4.5680e−11	5.5512e−17	1.9209e−24	4.4062e−31
1000	5000	1	1.6393e-09	4.0516e-15	6.6174e-24	1.2334e-31
		2	1.0585e-10	8.8818e-16	1.1410e-24	1.2334e-31
		3	2.3497e-10	2.2204e-16	5.8720e-24	1.5674e-30
		4	1.9304e-10	2.2204e-16	4.2234e-26	8.9581e-33
		5	1.9763e-10	2.2204e-16	7.5502e-24	1.7044e-30
		6	1.9294e-10	2.2204e-16	7.2269e-26	1.6033e-32
		7	1.9367e-10	2.2204e-16	1.8298e-24	4.0844e-31
		8	1.9347e-10	2.2204e-16	1.8962e-24	4.2284e-31
		9	1.9353e-10	2.2204e-16	1.8955e-24	4.2279e-31
		10	1.9351e-10	2.2204e-16	1.8530e-24	4.1327e-31

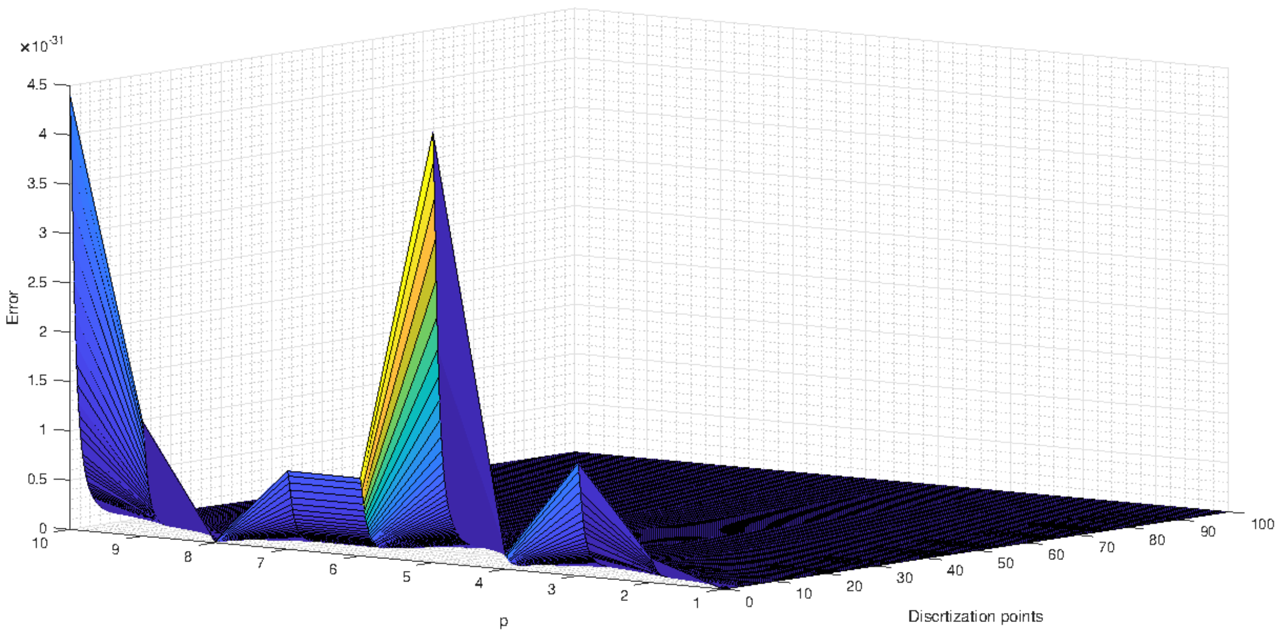

Figure 1. The error between the exact and iterative solution of equation (32) at $m = 600$

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Table 2. The error between the exact and approximate solution of equation (33) for different values of m, ν and p

b	m	p	ν		
			20	30	40
100	1000	1	0.0099	0.0018	2.7756e−17
		2	0.0374	4.3429e−04	1.5750e−12
		3	0.0340	8.9070e−04	4.1038e−12
		4	0.0215	4.0771e−04	3.0096e−12
		5	0.0269	4.7416e−04	3.8055e−12
		6	0.0234	2.3254e−04	3.6901e−12
		7	0.0251	4.1725e−04	3.7338e−12
		8	0.0242	3.9099e−04	3.7257e−12
		9	0.0245	4.0201e−04	3.7273e−12
		10	0.0244	3.9671e−04	3.7275e−12
500	20000	1	0.0715	0.0105	5.2112e−04
		2	0.1488	0.0033	2.4375e−05
		3	0.1778	0.0045	1.0036e−04
		4	0.1103	0.0017	2.2115e−05
		5	0.1553	0.0029	5.2616e−05
		6	0.1277	0.0021	3.3248e−05
		7	0.1435	0.0025	4.2169e−05
		8	0.1350	0.0024	3.9134e−05
		9	0.1367	0.0025	4.1402e−05
		10	0.1358	0.0025	4.0665e−05

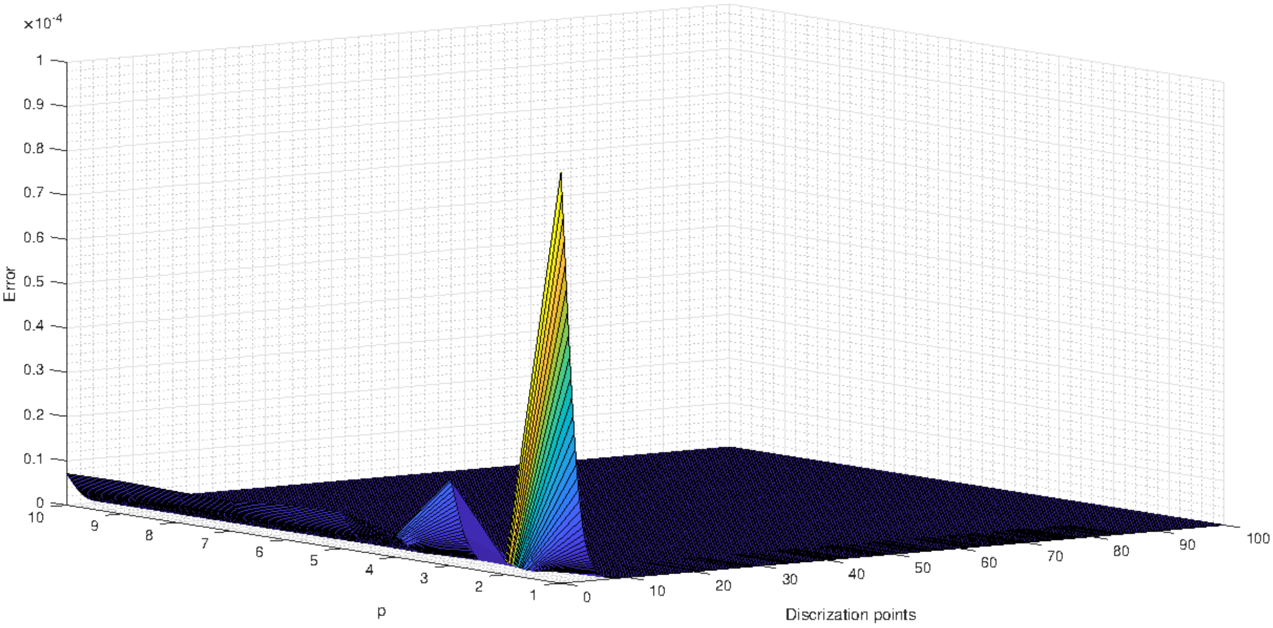

Figure 2. The error between the exact and iterative solution of equation (33) at $m = 600$

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Table 3. The error between the exact and approximate solution of equation (34) for different values of m, ν and p

a	m	p	ν		
			20	30	40
50	1000	1	0	0	0
		2	4.1203e-33	6.8363e-41	1.1343e-48
		3	1.0313e-32	2.9478e-40	8.4258e-48
		4	8.4401e-33	2.1573e-40	5.5142e-48
		5	8.8147e-33	2.3069e-40	6.0374e-48
		6	8.7334e-33	2.2743e-40	5.9226e-48
		7	8.7507e-33	2.2812e-40	5.9468e-48
		8	8.7470e-33	2.2797e-40	5.9416e-48
		9	8.7478e-33	2.2800e-40	5.9427e-48
		10	8.7476e-33	2.2800e-40	5.9425e-48
100	2000	1	0	0	0
		2	7.5177e-32	1.5918e-39	0
		3	8.3373e-32	2.0823e-39	3.3707e-47
		4	8.4650e-32	2.0815e-39	5.1184e-47
		5	8.4225e-32	2.0740e-39	5.1071e-47
		6	0.1277	8.4316e-32	5.1114e-47
		7	8.4299e-32	2.0756e-39	5.1105e-47
		8	8.4302e-32	2.0757e-39	5.1106e-47
		9	8.4301e-32	2.0756e-39	5.1106e-47
		10	8.4302e-32	2.0756e-39	5.1106e-47

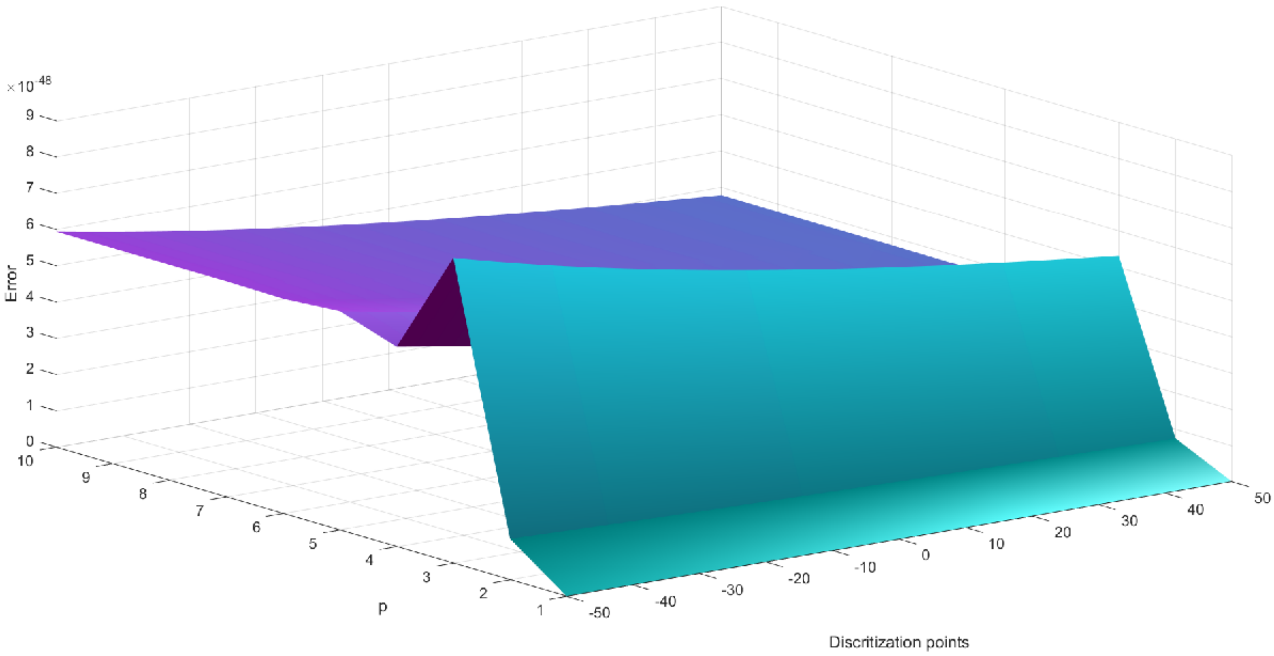

Figure 3. The error between the exact and iterative solution of equation (34) at $m = 1000$

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Tables 1, 2 and 3 show the values of errors between the exact and iterative solution, obtained in the iterative scheme (17) proposed in this paper, for three equations (32), (33) and (34). We consider the cases with two different values of b in each table and with two various values of the number of discretization m. Fixing these values, we vary the iterative number ν and p.

For instance, as it follows from table 3, we can stop our iterative procedure at $p = 1$. Meanwhile, the minimum of error is achieved at $p = 4$ in table 2. This means that in this case we can stop our iterative process at $p = 4$. Or, as can be seen from the table 1, we can go even further up to $p = 8$.

We conclude that in all cases we have examined, no matter how much we change the value of p, our proposed method (4) will remain convergent.

Using the Matlab program, we have plotted three figures 1, 2 and 3 demonstrating the solution error of three equations (32), (33) and (34) correspondingly as a function of the discretized points t_i and different values of p from $p = 1$ to $p = 10$.

7. Conclusion. In this paper we have searched for the best iterative scheme that can generalize two methods suggested by Atkinson in his work [22]. We have constructed such scheme using the same steps of Atkinson. Then, we have showed the convergence of the iterative solution u_m^ν to the Nyström solution u_m. We have proved the theorem on convergence of the iterative solution u_m^ν to the exact solution u in the Banach space $C^0[a, b]$. We have concluded our manuscript by presenting several numerical examples, whose purpose was to test the method we have proposed and to clarify the error behavior as a function of different values of p, ν, b and m.

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