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Two numerical treatments for solving the linear integro-differential Fredholm equation with a weakly singular kernel

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Abstract: We compare the error behavior of two methods used to find a numerical solution of the linear integro-differential Fredholm equation with a weakly singular kernel in Banach space $C^1[a,b]$. We construct an approximation solution based on the modified cubic b-spline collocation method. Another estimation of the exact solution, constructed by applying the numerical process of product and quadrature integration, is considered as well. Two proposed methods lead to solving a linear algebraic system. The stability and convergence of the cubic b-spline collocation estimate is proved. We test these methods on the concrete examples and compare the numerical results with the exact solution to show the efficiency and simplicity of the modified collocation method.

Keywords: singular integral equations, integro-differential equations, Fredholm integral equations.

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Два численных метода решения линейного интегро-дифференциального уравнения Фредгольма со слабо сингулярным ядром

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Аннотация: Мы сравниваем поведение ошибок двух методов, используемых для нахождения численного решения линейного интегро-дифференциального уравнения Фредгольма со слабо сингулярным ядром в банаховом пространстве $C^1[a,b]$. Мы строим приближенное решение на основе модифицированного кубического метода коллокации b-сплайнов. Рассматривается также другая оценка точного решения, построенная с применением численного процесса интегрирования по произведению и квадратурам. Два предложенных метода приводят к решению линейной алгебраической системы. Доказана устойчивость и сходимость кубической b-сплайновой коллокации. Мы тестируем эти методы на конкретном примере и сравниваем численные результаты с точным решением для того чтобы продемонстрировать эффективность и простоту модифицированного метода коллокации.

Ключевые слова: сингулярные интегральные уравнения, интегро-дифференциальные уравнения, интегральные уравнения Фредгольма.

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1. Introduction. Integro-differential equations are considered one of the most well-known mathematical equations, which are used in many fields, for instance in physics [1, 2], biology [3], dynamics [4, 5], medicine [6], computer science [7], etc. Their explicit form varies depending on the studied scientific task. A big number of these equations have been investigated previously in different papers on the relevant topics. Among them, the equations with a weakly singular kernel in the non-linear Volterra form [8–10], in the non-linear Fredholm form [11], in the linear Fredholm form [12] or in the non-linear Volterra-Fredholm form [13] play a prominent role. For example, they are extensively applied in the network studies [14, 15], in COVID-19 researches [16] and others.

Because of the great number of forms of these equations, the study of the analytical solution is difficult and generally impossible. However, nowadays the various numerical methods and techniques are known to find good approximations of their solutions, namely the domain decomposition method [1], the degenerate kernel [12], the product trapezoidal rule [8, 9, 11, 13, 17], the meshless local discrete collocation technique [18], the projection method [19] and the Fourier series [20].

In this manuscript we construct two different methods to search for an approximate solution of the linear integro-differential Fredholm equation with a weakly singular kernel, which we have already studied in the continuous case [21, 22]. This time, we put the singularity in the derivative part, such that the equation has the following form:

$$\forall x \in [a,b]: \quad \lambda u(x) = \int\limits_a^b K_1(x,t)u(t)dt + \int\limits_a^b p(|x-t|)K_2(x,t)u'(t)dt + f(x), \tag{1}$$

where $f(x)$ is a given term defined in $C^1[a,b]$ and λ is a real or complex parameter, which depends on physical quantities in practice.

In the first approach we construct a solution of Eq.(1) based on the b-spline collocation method [23–26]. We consider the error of this approximate method using the convergence of projection operator. In the second one, we apply the product integration method to transform our equations in a linear system containing 4 blocks. The convergence of the approximate solution is ensured by constructing a sufficient condition. At the end, we provide for the numerical examples to illustrate the difference between these two methods.

2. Preliminary. For a better understanding of the content of this manuscript by readers, we will start this section by introducing some functional basics and spaces.

Definition 1. *$C^0[a,b]$ is the Banach space of continuous functions $g(x)$ defined on $[a,b]$ to $\mathbb{R}$, with the following norm $\|.\|_\infty$*

$$\forall\, g \in C^0[a,b]: \quad \|g\|_\infty = \max_{a\leqslant x\leqslant b} |g(x)|\,.$$

Definition 2. *$C^1[a,b]$ is the Banach space of continuously differentiable functions $g(x)$, determined as*

$$C^1[a,b] := \Big\{ g:[a,b] \longrightarrow \mathbb{R}, \quad g, g' \in C^0[a,b] \Big\},$$

with the following norm

$$\forall\, g \in C^1[a,b]: \quad \|g\|_{C^1[a,b]} = \|g\|_\infty + \|g'\|_\infty.$$

Definition 3. *The Banach space $L^1(a,b)$ consists of equivalences classes of measurable functions g: $[a,b] \longrightarrow \mathbb{R}$, such that*

$$\|g\|_{L^1(a,b)} = \int\limits_a^b |g(x)|\,dx < \infty.$$

Definition 4. *The Sobolev space $W^{1,1}(a,b)$ is defined by*

$$W^{1,1}(a,b) := \Big\{ g \in L^1(a,b), \quad g' \in L^1(a,b) \Big\},$$

where g' is a weak derivative of g and $W^{1,1}(a,b)$ is a Banach space with the norm

$$\|g\|_{W^{1,1}[a,b]} = \|g\|_{L^1[a,b]} + \|g'\|_{L^1[a,b]}\,.$$

Definition 5. *Let X be the Banach space. The space $BL(X)$ is the Banach space of linear operators defined in X into itself which equipped with the following norm*

$$\forall M \in BL(X), \quad \|M\| = \sup_{\|v\|_X \leqslant 1} \|Mv\|_X.$$

For more details about the above mentioned spaces, see [27].

Further on, we need to define the continuity module κ_0 by: $\forall\, h > 0,\ \forall\, v \in C^0[a,b]$

$$\kappa_0(v,h) = \sup_{|x-y|\leqslant h} |v(x) - v(y)|,$$

and the continuity module $\kappa_1 : \forall\, h > 0,\ \forall v \in C^1[a,b]$

$$\kappa_1(v,h) = \kappa_0(v,h) + \kappa_0(v',h);$$

the continuity module $\kappa_{1,0}$ of any functions defined in the square $[a,b]^2$: $\forall x \in [a,b],\ \forall h > 0,\ \forall g \in C^0([a,b]^2, \mathbb{R})$,

$$\kappa_{1,0}(g,h)(x) = \sup_{|y_1-y_2|\leqslant h} |g(x,y_1) - g(x,y_2)|,$$

and the continuity module $\kappa_{1,1}$ as: $\forall x \in [a,b], \quad \forall h > 0, \quad \dfrac{\partial g}{\partial x} \in C^0\left([a,b]^2, \mathbb{R}\right)$

$$\kappa_{1,1}(g,h)(x) = \kappa_{1,0}(g,h)(x) + \kappa_{1,0}(\partial_x g, h)(x),$$

where $\partial_x g$ is a partial derivative of g with respect to x.

Also, we recall for any vector $v = (v_0, v_1, \ldots, v_n)^t \in \mathbb{R}^{n+1}$

$$\|v\|_{\mathbb{R}^{n+1}} = \max_{0\leqslant i\leqslant n} |v_i|,$$

and the matrix norm of $T = (t_{ij})_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$

$$|||\, T \,||| = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |t_{ij}|.$$

3. Problem Position. We assume that K_i in (1) satisfies the following hypothesis $(\mathcal{H}_1)$ for $i = 1, 2$

$$(\mathcal{H}_1) \left\| \begin{array}{l} \dfrac{\partial K_i}{\partial x}(x,t) \in C^0\left([a,b]^2, \mathbb{R}\right), \\ \exists M_i > 0, \quad \max\limits_{a\leqslant x,t\leqslant b} \left(|K_i(x,t)|\,,\ \left|\dfrac{\partial K_i}{\partial x}(x,t)\right| \right) \leqslant M_i. \end{array} \right.$$

and the singular part p meets the conjecture $(\mathcal{H}_2)$

$$(\mathcal{H}_2) \left\| \begin{array}{l} p \in W^{1,1}(0, b-a). \\ p(0) = 0, \\ \lim\limits_{s\to 0^+} |p'(s)| = +\infty. \end{array} \right.$$

Under the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, one can implicitly find the derivative u' :

$$\forall x \in [a,b] : \quad \lambda u'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t) u(t)\, dt + \int_a^b p(|x-t|) \frac{\partial K_2}{\partial x}(x,t) u'(t)\, dt + \\ + \int_a^b \mathrm{sign}(x-t) p'(|x-t|) K_2(x,t) u'(t)\, dt + f'(x), \qquad (2)$$

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where

$$\operatorname{sign}(x-t) = \begin{cases} 1, & \text{if} \quad x > t, \\ -1, & \text{else.} \end{cases}$$

4. Analytical Study. In this paper, we focus on the numerical solution of (1) by applying two different methods: the collocation b-spline and the product trapezoidal rule (or product integration). However, before numerically solving the integro-differential Fredholm equation, the existence and uniqueness of its solution must be verified. Below, we present the theorem that provides for these properties. But first, we introduce the linear operator A by:

$$\begin{aligned} A : C^1[a,b] &\longrightarrow C^1[a,b], \\ u &\longmapsto Au(x) = \int_a^b K_1(x,t)u(t)\,dt + \int_a^b p(|x-t|)K_2(x,t)u'(t)\,dt. \end{aligned} \tag{3}$$

The equation (1) is rewritten as

$$(\lambda I - A)u = f, \tag{4}$$

where I is the identity operator of $C^1[a,b]$.

Theorem 1. *If* $|\lambda| > 2\Big((b-a)M_1 + M_2 \,||\, p \,||_{W^{1,1}[0,b-a]}\Big)$, *then the equation* (4) *has a unique solution.*

Proof To prove that the equation (4) has a unique solution, we need to demonstrate that $(\lambda I - A)^{-1}$ exists and is bounded. Remind that the norm of the operator A reads

$$\|A\| = \sup_{\|u\|_{C^1[a,b]} \leqslant 1} \|Au\|_{C^1[a,b]}.$$

It is easy to proof that

$$\begin{aligned} |Au(x)| &\leqslant \big((b-a)M_1 + M_2\|p\|_{L^1[0,b-a]}\big)\, \|u\|_{C^1[a,b]}, \\ |(Au)'(x)| &\leqslant \big((b-a)M_1 + M_2\|p\|_{W^{1,1}[0,b-a]}\big)\, \|u\|_{C^1[a,b]}. \end{aligned}$$

In this case we obtain

$$\|A\| \leqslant 2\left((b-a)M_1 + M_2\|p\|_{W^{1,1}[0,b-a]}\right).$$

If $|\lambda| > 2((b-a)M_1 + M_2\|p\|_{W^{1,1}[0,b-a]})$, then $\|A\| < |\lambda|$. Accordingly to the Neumann's theorem [17], $(\lambda I - A)^{-1}$ exists and is bounded. This fact proves the statement of the theorem 1. For more details about the proof see [21, 22].

5. Numerical Methods. In many research papers devoted to the study of the integro-differential types of equations, an approximation solution based on the application of the product trapezoidal rule is constructed. This method of numerical solution is very useful in problems with singularity term. Recently, another class of the numerical methods was developed to look for a numerical solution of (1). It relies on the application of the b-spline approximations. The advantage of the b-spline collocation method is that it treats the equation (1) instead of the two equations (1)–(2), so we get an algebraic system of one block only.

These approaches are known for their simplicity and accuracy. In this section, we develop these two methods for Fredholm equation and study their convergence.

5.1. Collocation b-Spline Method. To use the collocation processes, primarily we need to give definition of the cubic b-spline functions (see [23, 25, 28–30]).

Definition 6. *Let* $\Delta_n = \{a = x_0 < x_1 < \ldots < x_n = b\}$ *be a uniform partition of interval* $[a,b]$ *with* $x_i = a + ih$ *and* $h = \dfrac{b-a}{n}$. *Let* $\mathcal{B}_3(\Delta_n)$ *be the space of cubic b-spline functions:*

$$\mathcal{B}_3(\Delta_n) = \left\{ S \in C^2[a,b] : \; S\big|_{[x_i,x_{i+1}]} \in P^3, \; i = 0, 1, \ldots, n-1 \right\},$$

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where $S\big|_{[x_i,x_{i+1}]}$ *is the restriction of the spline function* $S:[0,1]\longrightarrow\mathbb{R}$ *in each sub-interval* $[x_i,x_{i+1}]$ *and* P^3 *is the space of cubic polynomials. For* $i=-1,0,\dots,n,n+1$, *we define the following cubic B-spline:*

$$B_i^3(x)=\frac{1}{h^3}\begin{cases}(x-x_{i-2})^3, & x\in[x_{i-2},x_{i-1}],\\ (x-x_{i-2})^3-4(x-x_{i-1})^3, & x\in[x_{i-1},x_i],\\ (x_{i+2}-x)^3-4(x_{i+1}-x)^3, & x\in[x_i,x_{i+1}],\\ (x_{i+2}-x)^3, & x\in[x_{i+1},x_{i+2}],\\ 0, & \textit{otherwise}.\end{cases} \tag{5}$$

Here the values x_{-i} *at* $i=3,2,1$ *and* x_{n+1} *stand for* $x_{-i}=a-ih$ *and* $x_{n+1}=a+(n+1)h$ *correspondingly. It is clear that* $\{B_{-1},B_0,B_1,\dots,B_{n-1},B_n,B_{n+1}\}$ *forms a basis of* $\mathcal{B}_3(\Delta_n)$. *Then* $\mathcal{B}_3(\Delta_n)$ *is a finite dimensional linear sub-space of* $C^2[a,b]$ *with dimension* $n+3$.

We define $(P_n^3)_{n\in\mathbb{N}^*}$ as the sequence of linear projection operator

$$\begin{aligned}P_n^3: C^1[a,b]&\longrightarrow \mathcal{B}_3(\Delta_n)\\ v\ &\longmapsto P_n^3v(x)=\sum_{i=-1}^{n+1}\alpha_iB_i^3(x).\end{aligned} \tag{6}$$

It fulfills the following interpolation condition for all $v\in C^1[a,b]$

$$\begin{cases}P_n^3v(x_i)=v(x_i), & i=0,1,\dots,n,\\ (P_n^3v)'(a)=v'(a),\\ (P_n^3v)'(b)=v'(b).\end{cases} \tag{7}$$

We construct an approximation solution u_n, which satisfies the following equation:

$$\lambda u_n=P_n^3Au_n+P_n^3f. \tag{8}$$

§ 5.1.1. Convergence Analysis. In this part, we will demonstrate the convergence of P_n^3u to u in the sense of the norm $C^1[a,b]$. Previously, the authors of [31–36] have proven this convergence for the case when $u\in C^4[a,b]$. The convergence in the Banach space $C^0[-1,1]$ was considered in [37, 38], whereas the case of the periodic function u was investigated in [39]. To prove this convergence, we use the spline interpolation of u on the grid Δ_n with $n\geqslant 1$. In each interval $[x_{i-1},x_i]$ this spline $S_n(x)$ (see [23, 32, 34, 35]) is defined in the following way:

$$S_n(x)=M_{i-1}\frac{(x_i-x)^3}{6h}+M_i\frac{(x-x_{i-1})^3}{6h}+\left(u(x_{i-1})-\frac{M_{i-1}h^2}{6}\right)\left(\frac{x_i-x}{h}\right)+\\ +\left(u(x_i)-\frac{M_ih^2}{6}\right)\left(\frac{x-x_{i-1}}{h}\right), \tag{9}$$

where the moment M_i is $M_i=S''(x_i)$.

One should note that any cubic spline $S_n(x)$, constructed on segment $[a,b]$, is described by the linear combination of the cubic b-spline [33, 37, 40, 41]. In this regard, $S_n(x)$ will read:

$$\forall x\in[a,b]:\quad S_n(x)=P_n^3u(x). \tag{10}$$

This leads to the following equality:

$$\|(I-P_n^3)u\|_{C^1[a,b]}=\|u-S_n\|_{C^1[a,b]}. \tag{11}$$

The values of the derivatives of the spline $S_n(x)$ (9) over x in the left and right limit of the point x_i are

$$S_n'(x_i^-)=\frac{h}{6}M_{i-1}+\frac{h}{3}M_i+\frac{u(x_i)-u(x_{i-1})}{h}, \tag{12}$$

$$S_n'(x_i^+)=-\frac{h}{3}M_i-\frac{h}{6}M_{i+1}+\frac{u(x_{i+1})-u(x_i)}{h}. \tag{13}$$

The continuity of $S_n'(x)$ at x_i yields for $i = 1, 2, \dots, n-1$

$$M_{i-1} + 4M_i + M_{i+1} = 6\left(\frac{u(x_{i+1}) - u(x_i)}{h^2} - \frac{u(x_i) - u(x_{i-1})}{h^2}\right). \tag{14}$$

However, for different applications it is more convenient to work with the slopes $m_i = S_n'(x_i)$ rather than the moments M_i. Below we present another representation of S_n and S_n' in each segment $[x_{i-1}, x_i]$:

$$S_n(x) = m_{i-1}\frac{(x_i - x)^2(x - x_{i-1})}{h^2} - m_i\frac{(x - x_{i-1})^2(x_i - x)}{h^2} + \\ + u(x_{i-1})\frac{(x_i - x)^2[2(x - x_{i-1}) + h]}{h^2} + u(x_i)\frac{(x - x_{i-1})^2[2(x_i - x) + h]}{h^2}, \tag{15}$$

$$S_n'(x) = m_{i-1}\frac{(x_i - x)(2x_{i-1} + x_i - 3x)}{h^2} - m_i\frac{(x - x_{i-1})(2x_i + x_{i-1} - 3x)}{h^2} + \\ + \frac{u(x_i) - u(x_{i-1})}{h^3}6(x_i - x)(x - x_{i-1}). \tag{16}$$

The limit values of the second derivative of $S(x)$ at x_i are equal to

$$S''(x_i^-) = \frac{2}{h}m_{i-1} + \frac{4}{h}m_i - 6\frac{u(x_i) - u(x_{i-1})}{h^2}, \tag{17}$$

$$S''(x_i^+) = -\frac{4}{h}m_i - \frac{2}{h}m_{i+1} + 6\frac{u(x_{i+1}) - u(x_i)}{h^2}. \tag{18}$$

Since $S''(x)$ is continuous, we get that

$$m_{i-1} + 4m_i + m_{i+1} = 3\frac{u(x_{i+1}) - u(x_{i-1})}{h}, \quad i = 1, 2, \dots, n-1. \tag{19}$$

To determine all $(n + 1)$ quantities including in (19), we should add two extra relations following from the boundary conditions. Using the interpolation conditions in (7) and the equation (10), we find that

$$S_n'(a) = u'(x_0) \quad \text{and} \quad S_n'(b) = u'(x_n).$$

Now, accommodating the equation (3), we obtain the following relation:

$$2M_0 + M_1 = \frac{6}{h}\left[\frac{u(x_1) - u(x_0)}{h} - u'(x_0)\right].$$

In the similar way, using (2), we get

$$M_{n-1} + 2M_n = \frac{6}{h}\left[u'(x_n) - \frac{u(x_n) - u(x_{n-1})}{h}\right].$$

The recurrent relations (14) and (19) can be written in the matrix forms $CM = E$ and $Cm = D$ correspondingly with matrix C of size $(n + 1) \times (n + 1)$

$$C = \begin{bmatrix} 2 & 1 & & \dots\dots\dots & & & 0 \\ 1 & 4 & 1 & \dots\dots\dots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots\dots\dots & & & 1 & 4 & 1 \\ 0 & \dots\dots\dots\dots & & & & 2 & 1 \end{bmatrix},$$

and $M = (M_0, M_1, \dots, M_n)^t \in \mathbb{R}^{n+1}$, $m = (m_0, m_1, \dots m_n)^t \in \mathbb{R}^{n+1}$. The vector $D = (d_0, d_1, \dots d_n)^t \in \mathbb{R}^{n+1}$ has the following components

$$\begin{cases} d_0 &= 3\,\dfrac{u(x_1) - u(x_0)}{h}, \\ d_i &= 3\,\dfrac{u(x_{i+1}) - u(x_{i-1})}{h}, \quad i = 1, 2, \dots, n-1, \\ d_n &= 3\,\dfrac{u(x_{n-1}) - u(x_n)}{h}, \end{cases} \tag{20}$$

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and the elements of the vector $E = (e_0, e_1, \dots, e_n)^t \in \mathbb{R}^{n+1}$ read

$$\begin{cases} e_0 &= \dfrac{6}{h}\left(\dfrac{u(x_1)-u(x_0)}{h} - u'(x_0)\right), \\ e_i &= 6\left(\dfrac{u(x_{i+1})-u(x_i)}{h^2} - \dfrac{u(x_i)-u(x_{i-1})}{h^2}\right), \quad i = 1, 2, \dots n-1, \\ e_n &= \dfrac{6}{h}\left(u'(x_n) - \dfrac{u(x_n)-u(x_{n-1})}{h}\right). \end{cases} \tag{21}$$

Theorem 2. *Let $P_n^3 u$ is defined by (6) and u is the exact solution of (1), then*

$$\|u - P_n^3 u\|_{C^1[a,b]} \leqslant c\kappa_1(u, h), \tag{22}$$

where c is a positive constant.

Proof Using the spline formula (9) in each interval $[x_{i-1}, x_i]$, we get

$$P_n^3 u(x) - u(x) = S_n(x) - u(x) = \frac{(x-x_i)(x-x_{i-1})}{6h}\Big[(2x_i - x_{i-1} - x)M_{i-1} + (x - 2x_{i-1} + x_i)M_i\Big] + \\ + \left(\frac{u(x_i)+u(x_{i-1})}{2} - f(x)\right) - \Big(u(x_i) - u(x_{i-1})\Big)\frac{x_i + x_{i-1} - 2x}{2h}. \tag{23}$$

If we demand that the following inequalities are fulfilled

$$\begin{aligned} (x_i - x)(x - x_{i-1}) &\leqslant h^2, \\ (2x_i - x_{i-1} - x) &\leqslant 2h, \\ (x - 2x_{i-1} + x_i) &\leqslant 2h, \\ |x_i + x_{i-1} - 2x| &\leqslant h, \end{aligned}$$

then, we obtain

$$|S_n(x) - u(x)| \leqslant \frac{h^2}{3}\Big[\,|M_{i-1}| + |M_i|\,\Big] + \left|\frac{u(x_i)-u(x_{i-1})}{2} - u(x)\right| + \frac{|u(x_i)+u(x_{i-1})|}{2}.$$

This leads us to the relation

$$\|S_n - u\|_\infty \leqslant \frac{2h^2}{3}\|M\|_{\mathbb{R}^{n+1}} + \frac{3}{2}\,\kappa_0(u, h). \tag{24}$$

On the other hand, we have $M = C^{-1}E$ and $\|M\|_{\mathbb{R}^{n+1}} \leqslant |||\ C^{-1}\ |||\ \|E\|_{\mathbb{R}^{n+1}}$, where

$$\|E\|_{\mathbb{R}^{n+1}} = \max_{0\leqslant i\leqslant n} |e_i| = \frac{6}{h^2}\max_{0\leqslant i\leqslant n}\left|\frac{u(x_{i+1})-u(x_i)}{h} - \frac{u(x_i)-u(x_{i-1})}{h}\right| \leqslant \frac{3}{h^2}\kappa_0(u, h). \tag{25}$$

Thus,

$$\|S_n - u\|_\infty \leqslant \left(2|||\ C^{-1}\ ||| + \frac{3}{2}\right)\kappa_0(u, h). \tag{26}$$

Now, applying the interpolation formula (16) in each segment $[x_{i-1}, x_i]$, we find

$$S'_n(x) - \frac{u(x_i)-u(x_{i-1})}{h} = \left[\frac{3}{h^2}\left(x - \frac{x_{i-1}+x_i}{2}\right)^2 - \frac{1}{4}\right]\Big[m_{i-1} - u'(x_{i-1}) + m_i - u'(x_i) + u(x_i) + u(x_{i-1}) + \\ -2\left(\frac{u(x_i)-u(x_{i-1})}{h}\right)\Big] + \frac{1}{h}\left(x - \frac{x_i + x_{i-1}}{2}\right)\Big[m_i - u'(x_i) + m_{i-1} - u'(x_{i-1}) + u'(x_i) + u'(x_{i-1})\Big]. \tag{27}$$

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Here we have used the relations:

$$\frac{6}{h^2}(x-x_{i-1})(x_i-x) = -\frac{6}{h^2}\left(x-\frac{x_i-x_{i-1}}{2}\right)^2+\frac{3}{2},$$
$$\frac{1}{h}(x_i-x)(2x_{i-1}+x_i-3x) = \frac{3}{h^2}\left(x-\frac{x_{i-1}+x_i}{2}\right)^2-\frac{1}{4}+\frac{1}{h}\left(x-\frac{x_i+x_{i-1}}{2}\right).$$

Taking into account the expressions, presented above, one can get

$$\left|S_n'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\right| \leqslant \frac{1}{2}\Big[|m_{i-1}-u'(x_{i-1})|+|m_i-u'(x_i)|+2\kappa_0(u',h)\Big]+ \\ +|m_i-u'(x_i)|+|m_{i-1}-u'(x_{i-1})|+\kappa_0(u',h). \tag{28}$$

Let us now write down the following obvious equality

$$C\left(m-\frac{D}{6}\right)=\left(I_{n+1}-\frac{C}{6}\right)D, \tag{29}$$

where I_{n+1} is the identity matrix of size $(n+1)\times(n+1)$. The r.h.s. of (29) is equal to:

$$\left(I_{n+1}-\frac{1}{6}C\right)D = \begin{bmatrix} \dfrac{d_0}{3}-\dfrac{d_1}{6} \\[2ex] \dfrac{-d_0}{6}+\dfrac{d_1}{3}-\dfrac{d_2}{6} \\ \vdots \\ \dfrac{-d_{n-2}}{6}+\dfrac{d_{n-1}}{3}-\dfrac{d_n}{6} \\[2ex] \dfrac{-d_{n-1}}{6}+\dfrac{d_n}{3} \end{bmatrix}. \tag{30}$$

Using the definition of the vector D (20) and the condition

$$\left|u'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\right| \leqslant \kappa_0(u',h), \tag{31}$$

we obtain the following result

$$\begin{cases} \left|\dfrac{d_0}{3}-\dfrac{d_1}{6}\right| \leqslant 3\kappa_0(u',h), \\ \left|\dfrac{-d_{i-1}}{6}+\dfrac{d_i}{3}-\dfrac{d_{i+1}}{6}\right| \leqslant 6\kappa_0(u',h), \quad i=1,2,\dots,n-1, \\ \left|\dfrac{-d_{n-1}}{6}+\dfrac{d_n}{3}\right| \leqslant 3\kappa_0(u',h). \end{cases} \tag{32}$$

The inequalities (32) enable to obtain that $\left\|\left(I_{n+1}-\dfrac{C}{6}\right)D\right\|_{\mathbb{R}^{n+1}} \leqslant 6\ \kappa_0(u',h)$. In this case, it follows from (29):

$$\left\|m-\frac{D}{6}\right\|_{\mathbb{R}^{n+1}} \leqslant 6|||\ C^{-1}\ |||\ \kappa_0(u',h). \tag{33}$$

Substituting (33) in (8), we arrive to the inequalities

$$\left|S_n'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\right| \leqslant (11|||\ C^{-1}\ |||+1)\kappa_0(u',h). \tag{34}$$

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It is evident from (34) and (31) that

$$\|S_n' - u'\|_\infty \leqslant \max_{a\leqslant x\leqslant b}\left|S_n'(x) - \frac{u(x_i)-u(x_{i-1})}{h}\right| + \max_{a\leqslant x\leqslant b}\left|\frac{u(x_i)-u(x_{i-1})}{h} - u'(x)\right| \leqslant$$
$$\leqslant \big(11|||\ C^{-1}\ ||| + 2\big)\,\kappa_0(u',h). \tag{35}$$

Combining (26) and (35), we obtain

$$\|u - P_n^3 u\|_{C^1[a,b]} = \|u - S_n\|_{C^1[a,b]} \leqslant c\,\kappa_1(u,h), \tag{36}$$

where $c = \max\left\{11|||\ C^{-1}\ ||| + 1;\ 2|||\ C^{-1}\ ||| + \dfrac{3}{2}\right\}$.

Theorem 3. *Let P_n^3 be a projection operator given by* (6) *and A is a compact operator defined by* (3), *then*

$$\lim_{n\to\infty}\|(I - P_n^3)A\| = 0. \tag{37}$$

Proof Since A is compact, then the set $\mathcal{M} = \left\{Au,\ u\in C^1[a,b],\ \|u\|_{C^1[a,b]}\leqslant 1\right\}$ is relatively compact in the Banach space $C^1[a,b]$ and by Banach–Steinhaus theorem's [19, 42], P_n^3 converges uniformly to the identity operator I in $C^1[a,b]$:

$$\lim_{n\to\infty}\|(I-P_n^3)A\| = \lim_{n\to\infty}\sup_{\|u\|_{C^1[a,b]}}\|(I-P_n^3)Au\|_{C^1[a,b]} = \sup_{v\in\mathcal{M}}\|(I-P_n^3)v\|_{C^1[a,b]}.$$

Under above theorems, we get that P_n^3 is pointwise convergent to the identity operator and $P_n^3 A$ converges to A, then $(\lambda I - P_n^3 A)^{-1}$ exists and $\|(\lambda I - P_n^3 A)^{-1}\| < \infty$ (see [17]).

Theorem 4. *Let u_n be the solution of* (8) *and u be the exact solution of* (4), *then*

$$\lim_{n\to\infty}\|u - u_n\|_{C^1[a,b]} = 0 \tag{38}$$

Proof For large n enough, $u - u_n = (\lambda I - P_n^3 A)^{-1}\left[(A - P_n^3 A)u + (f - P_n^3 f)\right]$. Then, the following estimation is valid:

$$\|u - u_n\|_{C^1[a,b]} \leqslant \|(\lambda I - P_n^3 A)^{-1}\|\cdot\left[\left\|(I-P_n^3)A\right\| + \left\|(I-P_n^3)f\right\|_{C^1[a,b]}\right]. \tag{39}$$

When $n\to\infty$, we get to the desired result.

§ 5.1.2. System Approximation. For all $x\in[a,b]$ it follows from the equation (8) that

$$\lambda u_n(x) = P_n^3 A u_n(x) + P_n^3 f(x). \tag{40}$$

We put $x = x_j$ for $j = 0,1,\ldots,n$. Then by the interpolation conditions of the sequence $(P_n^3)_{n\in\mathbb{N}}$ (7), we get

$$\lambda u_n(x_j) = A u_n(x_j) + f(x_j), \tag{41}$$

which is equivalent to the following algebraic system for $j = 0,1,\ldots,n$

$$\sum_{i=-1}^{n+1}\alpha_i\left[\lambda B_i^3(x_j) - \int_a^b K_1(x_j,t)B_i^3(t)\,dt - \int_a^b p\left(|x_j - t|\right)\,K_2(x_j,t)(B_i^3(t))'\,dt\right] = f(x_j), \tag{42}$$

where $\{\alpha_i\}_{i=-1}^{n+1}$ are unknowns coefficients to be determined.

Thus, we arrive to the system (42) containing $n+1$ equations with $n+3$ unknowns α_i. To handle this problem, we introduce $\overline{B^3}_i(x)$ as a modified basis of cubic b-splines [28]

$$\begin{cases} \overline{B^3}_0(x) = B_0^3(x) + 2B_{-1}^3(x), & \text{for } j = 0,\\ \overline{B^3}_1(x) = B_1^3(x) - B_{-1}^3(x), & \text{for } j = 1,\\ \overline{B^3}_j(x) = B_j^3(x), & \text{for } j = 2,\ldots,n-2,\\ \overline{B^3}_{n-1}(x) = B_{n-1}^3(x) - B_{n+1}^3(x), & \text{for } j = n-1,\\ \overline{B^3}_n(x) = B_n^3(x) + 2B_{n+1}^3(x), & \text{for } j = n. \end{cases} \tag{43}$$

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Table 1. The values of the modified b-cubic spline and its derivatives

x_j	$j=0$	$j=i-2$	$j=i-1$	$j=i$	$j=i+1$	$j=i+2$	$j=n$
$\overline{B^3}_i(x_j)$	6	0	1	4	1	0	6
$(\overline{B^3}_i(x_j))'$	$-\frac{6}{h}$	0	$-\frac{3}{h}$	0	$\frac{3}{h}$	0	$\frac{6}{h}$

In the table 1, we present the values of $\overline{B^3}_i(x_j)$ and its derivatives.

Finally, the solution u_n will have a new representation

$$\forall x \in [a,b]: \quad u_n(x) = \sum_{i=0}^{n} \overline{\alpha}_i \overline{B}_i^3(x), \tag{44}$$

and the system (42) will have a new form

$$\sum_{i=0}^{n} \overline{\alpha_i} \left[\lambda \overline{B}_i^3(x_j) - \int_a^b K_1(x_j,t)\overline{B}_i^3(t)\,dt - \int_a^b p\,(|x_j - t|)\; K_2(x_j,t)(\overline{B_i^3}(t))'\,dt \right] = f(x_j), \tag{45}$$

where $\{\overline{\alpha}_i\}_{i=0}^n$ are new $(n+1)$ coefficients to be determined.

5.2. Product integration. In this section, we apply the product integration method to the equations (1)–(2). First, we need to define the uniform partition Δ_n of the interval $[a,b]$ by

$$\forall n \geqslant 1, \quad \Delta_n = \left\{ x_i = a + ih,\; h = \frac{b-a}{n},\; i = 0, 1 \ldots, n \right\}.$$

There are two parts in equations (1) and (2), namely the regular and the weakly singular. Therefore, we utilize two methods to construct a numerical solution of (1)–(2). The classical Nyström [19, 43] method is used by us to treat the regular part. For that we employ the following approximation:

$$\forall g \in C^0[a,b]: \quad \int_a^b g(x)dx \simeq \sum_{i=0}^{n} \omega_i g(x_i), \tag{46}$$

where $\{\omega_i\}_{i=0}^n$ are called the weights, which we choose in accordance to the trapezoidal rule

$$h = \frac{b-a}{n}, \quad \omega_0 = \omega_n = \frac{h}{2}, \quad \omega_1 = \omega_2 = \ldots = \omega_{n-1} = h,$$
$$x_i = a + (i-1)h,\; i = 0, 1, \ldots, n, \quad \text{where} \quad \sum_{i=0}^{n} \omega_i = b - a.$$

To deal with the weakly singular part, we apply the product trapezoidal rule [17, 42]. For all $L \in C^0([a,b]^2, \mathbb{R})$, $v \in C^0[a,b]$ and $\forall n \geqslant 1,\ \forall t \in [a,b]:$

$$\big[L(x,t)v(t)\big]_n = \frac{1}{h}\big[(x_i - t)L(x, x_{i-1})v(x_{i-1}) + (t - x_{i-1})L(x, x_i)v(x_i)\big]. \tag{47}$$

Using the described above two numerical schemes (46) and (47), we get the following system

$$\begin{cases} \lambda u(x) = \sum_{i=0}^{n} \omega_i K_1(x,x_i)u(x_i) + \sum_{i=0}^{n} \bar{\omega}_{1,i}(x) K_2(x,x_i)u'(x_i) + f(x) + \varepsilon_{1,n}(x) + \bar{\varepsilon}_{1,n}(x), \\ \lambda u'(x) = \sum_{i=0}^{n} \omega_i \frac{\partial K_1}{\partial x}(x,x_i)u(x_i) + \sum_{i=0}^{n} \left[\bar{\omega}_{1,i}(x)\frac{\partial K_2}{\partial x}(x,x_i) + \bar{\omega}_{2,i}(x)K_2(x,x_i)u'(x_i) \right] + \\ \qquad + f'(x) + \varepsilon_{2,n}(x) + \bar{\varepsilon}_{2,n}(x) + \bar{\varepsilon}_{3,n}(x), \end{cases} \tag{48}$$

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where

$$\begin{cases} \bar{\omega}_{1,0}(x) = \dfrac{1}{h}\displaystyle\int_a^{x_1} (x_1 - t)p(|x-t|)\,dt, \\ \bar{\omega}_{1,n}(x) = \dfrac{1}{h}\displaystyle\int_{x_{n-1}}^{x_n} (t - x_{n-1})p(|x-t|)\,dt, \\ \bar{\omega}_{1,i}(x) = \dfrac{1}{h}\displaystyle\int_{x_{i-1}}^{x_i} (t - x_{i-1})p(|x-t|)\,dt + \dfrac{1}{h}\displaystyle\int_{x_i}^{x_{i-1}} (x_{i+1} - t)p(|x-t|)\,dt, \quad i = 1, \dots, n-1, \end{cases}$$

and

$$\begin{cases} \bar{\omega}_{2,0}(x) = \dfrac{1}{h}\displaystyle\int_a^{x_1} \operatorname{sign}(x-t)(x_1 - t)p'(|x-t|)\,dt, \\ \bar{\omega}_{2,n}(x) = \dfrac{1}{h}\displaystyle\int_{x_{n-1}}^{x_n} \operatorname{sign}(x-t)(t - x_{n-1})p'(|x-t|)\,dt, \\ \bar{\omega}_{2,i}(x) = \dfrac{1}{h}\displaystyle\int_{x_{i-1}}^{x_i} \operatorname{sign}(x-t)(t - x_{i-1})p'(|x-t|)\,dt + \dfrac{1}{h}\displaystyle\int_{x_i}^{x_{i-1}} \operatorname{sign}(x-t)(x_{i+1} - t)p'(|x-t|)\,dt, \\ \qquad i = 1, \dots, n-1. \end{cases}$$

The local errors $\{\varepsilon_{p,n}\}$ with $p = 1, 2$ and $\{\bar{\varepsilon}_{p,n}\}$ with $p = 1, 2, 3$ at $\forall n \geqslant 1$, included in (48), read

$$\varepsilon_{1,n}(x) = \int_a^b K_1(x,t)u(t)\,dt - \sum_{i=0}^{n} \omega_i K_1(x, x_i)u(x_i), \tag{49}$$

$$\bar{\varepsilon}_{1,n}(x) = \int_a^b p(|x-t|)K_2(x,t)u'(t)\,dt - \sum_{i=0}^{n} \bar{\omega}_{1,i}(x)K_2(x, x_i)u'(x_i), \tag{50}$$

$$\varepsilon_{2,n}(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t)\,dt - \sum_{i=0}^{n} \omega_i \frac{\partial K_1}{\partial x}(x, x_i)u(x_i), \tag{51}$$

$$\bar{\varepsilon}_{2,n}(x) = \int_a^b p(|x-t|)\frac{\partial K_2}{\partial x}(x,t)u'(t)\,dt - \sum_{i=0}^{n} \bar{\omega}_{1,i}(x)\frac{\partial K_2}{\partial x}(x, x_i)u'(x_i), \tag{52}$$

$$\bar{\varepsilon}_{3,n}(x) = \int_a^b \operatorname{sign}(x-t)p'(|x-t|)K_2(x,t)u'(t)dt - \sum_{i=0}^{n} \bar{\omega}_{2,i}(x)K_2(x, x_i)u'(x_i). \tag{53}$$

§ 5.2.1. System Approximation. We put $x = x_j$ and assume that the presented errors (49)–(53) are negligible in (48). Then we get the following linear system with $2n+2$ equations. For $0 \leqslant j \leqslant n$

$$\begin{cases} \lambda u_j = \displaystyle\sum_{i=0}^{n} \omega_i K_1(x_j, x_i)u_i + \sum_{i=0}^{n} \bar{\omega}_{1,i}(x_j)K_2(x_j, x_i)u'_i + f_j, \\ \lambda u'_j = \displaystyle\sum_{i=0}^{n} \omega_i \frac{\partial K_1}{\partial x}(x_j, x_i)u_i + \sum_{i=0}^{n}\left[\bar{\omega}_{1,i}(x_j)\frac{\partial K_2}{\partial x}(x_j, x_i) + \bar{\omega}_{2,i}(x_j)K_2(x_j, x_i)\right]u'_i + f'_j, \end{cases} \tag{54}$$

where $f_j = f(x_j)$, $f'_j = f'(x_j)$ and u_j, u'_j are the approximations of $u(x_j)$ and $u'(x_j)$ respectively.

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Theorem 5. *If* $2\Big[(b-a)M_1 + \|p\|_{W^{11}(0,b-a)}M_2\Big] <\mid \lambda \mid$, *then there exists a unique vector*

$$U = (u_0, u_1, \dots u_n, u'_0, u'_1, \dots u'_n)^t \in \mathbb{R}^{2n+2}$$

with the norm

$$\|U\|_{\mathbb{R}^{2n+2}} = \max_{0\leqslant i\leqslant n} \mid u_i \mid + \max_{0\leqslant i\leqslant n} \mid u'_i \mid,$$

that is the solution of the system (54).

Proof We use the Banach's fixed-point theorem [17] to prove the solution's existence and uniqueness of the system (54). It can be rewritten in the following form

$$U = T[U],$$

where $T[U]$ is a vector of $\mathbb{R}^{2n+2}$ given as

$$T[U] = \begin{cases} \dfrac{1}{\lambda}\displaystyle\sum_{i=0}^{n}\omega_i K_1(x_j, x_i)u_i + \dfrac{1}{\lambda}\displaystyle\sum_{i=0}^{n}\bar{\omega}_{1,i}(x_j)K_2(x_j, x_i)u'_i + f_j, \quad 0 \leqslant j \leqslant n, \\ \dfrac{1}{\lambda}\displaystyle\sum_{i=0}^{n}\omega_i \dfrac{\partial K_1}{\partial x}(x_{j-n-1}, x_i)u_i + \dfrac{1}{\lambda}\displaystyle\sum_{i=0}^{n}\Big[\bar{\omega}_{1,i}(x_j)\dfrac{\partial K_2}{\partial x}(x_{j-n-1}, x_i) + \\ \qquad + \bar{\omega}_{2,i}(x_j)K_2(x_{j-n-1}, x_i)\Big]u'_i + f'_{j-n-1}, \quad n+1 \leqslant j \leqslant 2n+2. \end{cases} \tag{55}$$

Let U and V are two vectors of $\mathbb{R}^{2n+2}$. Since

$$\left|\frac{1}{\lambda}\sum_{i=0}^{n}\omega_i K_1(x_j, x_i)(u_i - v_i) + \frac{1}{\lambda}\sum_{i=0}^{n}\bar{\omega}_{1,i}(x_j)K_2(x_j, x_i)(u'_i - v'_i)\right| \leqslant \frac{1}{|\lambda|}\left|\sum_{i=0}^{n}\omega_i\right| \, |K_1(x_j, x_i)| \max_{0\leqslant i\leqslant n} \mid u_i - v_i \mid + \\ + \frac{1}{|\lambda|}\left|\sum_{i=0}^{n}\bar{\omega}_{1,i}(x_j)\right| \, |K_2(x_j, x_i)| \max_{0\leqslant i\leqslant n} \mid u'_i - v'_i \mid$$

and

$$\left|\frac{1}{\lambda}\sum_{i=0}^{n}\omega_i \frac{\partial K_1}{\partial x}(x_{j-n-1}, x_i)(u_i - v_i) + \frac{1}{\lambda}\sum_{i=0}^{n}\Big[\bar{\omega}_{1,i}(x_j)\frac{\partial K_2}{\partial x}(x_{j-n-1}, x_i) + \bar{\omega}_{2,i}(x_j)K_2(x_{j-n-1}, x_i)\Big](u'_i - v'_i)\right| \leqslant \\ \leqslant \frac{1}{|\lambda|}\left|\sum_{i=0}^{n}\omega_i\right| \, \left|\frac{\partial K_1}{\partial x}(x_j, x_i)\right| \max_{0\leqslant i\leqslant n} \mid u_i - v_i \mid + \frac{1}{|\lambda|}\left|\sum_{i=0}^{n}\bar{\omega}_{1,i}(x_j)\right| \, \left|\frac{\partial K_2}{\partial x}(x_j, x_i)\right| \max_{0\leqslant i\leqslant n} \mid u'_i - v'_i \mid + \\ + \frac{1}{|\lambda|}\left|\sum_{i=0}^{n}\bar{\omega}_{2,i}(x_j)\right| \, \left|K_2(x_j, x_i)\right| \max_{0\leqslant i\leqslant n} \mid u'_i - v'_i \mid,$$

where $\left|\displaystyle\sum_{i=0}^{n}\omega_i\right| = b - a$, $\left|\displaystyle\sum_{i=0}^{n}\bar{\omega}_{1,i}(x_j)\right| \leqslant ||p||_{L^1(0,b-a)}$ and $\left|\displaystyle\sum_{i=0}^{n}\bar{\omega}_{2,i}(x_j)\right| \leqslant ||p'||_{L^1(0,b-a)}$, then

$$\|T[U] - T[V]\|_{\mathbb{R}^{2n+2}} \leqslant \frac{2(b-a)M_1}{|\lambda|}\max_{0\leqslant i\leqslant n} \mid u_i - v_i \mid + \frac{2M_2||p||_{W^{11}(0,b-a)}}{|\lambda|}\max_{0\leqslant i\leqslant n} \mid u'_i - v'_i \mid. \tag{56}$$

This means that

$$\|T[U] - T[V]\|_{\mathbb{R}^{2n+2}} \leqslant 2\,\frac{(b-a)M_1 + \|p\|_{W^{11}(0,b-a)}M_2}{\mid \lambda \mid}\|U - V\|_{\mathbb{R}^{2n+2}}. \tag{57}$$

Assuming the validity of the inequality $2\Big[(b-a)M_1 + \|p\|_{W^{11}(0,b-a)}M_2\Big] < \mid \lambda \mid$ and using the Banach's fixed-point theorem, we conclude that the approximate system has a unique solution.

§ 5.2.2. Convergence Analysis. The convergence analysis of this method is totally different from the first one because in this case we use the numerical techniques to prove that the errors (49)–(53) tend to zero. We start with the following theorems, which explain this decrease of local errors.

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Theorem 6. *Let $n \geqslant 1$ and ε_n be a vector of $\mathbb{R}^{2n+2}$*

$$\varepsilon_n = \big(\varepsilon_{1,n}(x_0), \varepsilon_{1,n}(x_1), \ldots, \varepsilon_{1,n}(x_n), \varepsilon_{2,n}(x_0), \varepsilon_{2,n}(x_1), \ldots, \varepsilon_{2,n}(x_n)\big)^t,$$

then

$$\|\varepsilon_n\|_{\mathbb{R}^{2n+2}} \leqslant (b-a)\left[\max_{0\leqslant i\leqslant n} \kappa_{1,1}(K_1,h)(x_i)\|u\|_{C^1[a,b]} + 2M_1\ \kappa_1(u,h)\right].$$

Proof For $i = 0, 1, \ldots, n$ we have

$$\big|\varepsilon_{1,n}(x_i)\big| = \left|\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} K_1(x_i,t)u(t)\,dt - \frac{h}{2}\Big[K_1(x_i,x_{j+1})u(x_{j+1}) + K_1(x_i,x_j)u(x_j)\Big]\right|,$$

$$\big|\varepsilon_{2,n}(x_i)\big| = \left|\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \frac{\partial K_1}{\partial x}(x_i,t)u(t)\,dt - \frac{h}{2}\left[\frac{\partial K_1}{\partial x}(x_i,x_{j+1})u(x_{j+1}) + \frac{\partial K_1}{\partial x}(x_i,x_j)u(x_j)\right]\right|.$$

But for all $t \in [x_j, x_{j+1}]$,

$$\left|\int_{x_j}^{x_{j+1}} K_1(x_i,t)u(t)\,dt - \frac{h}{2}\big[K_1(x_i,x_{j+1})u(x_{j+1}) + K_1(x_i,x_j)u(t_j)\big]\right| \leqslant$$
$$\leqslant h\left[\max_{0\leqslant i\leqslant n}\kappa_{1,0}(K_1,h)(x_i)\|u\|_{C^1[a,b]} + M_1\kappa_0(u,h)\right], \qquad (58)$$

$$\left|\int_{x_j}^{x_{j+1}} \frac{\partial K_1}{\partial x}(x_i,t)u(t)\,dt - \frac{h}{2}\left[\frac{\partial K_1}{\partial x}(x_i,x_{j+1})u(x_{j+1}) + \frac{\partial K_1}{\partial x}(x_i,x_j)u(x_j)\right]\right| \leqslant$$
$$\leqslant h\left[\max_{0\leqslant i\leqslant n}\kappa_{1,0}(\partial_x K_1,h)(x_i)\|u\|_{C^1[a,b]} + M_1\kappa_0(u,h)\right]. \qquad (59)$$

This leads us to:

$$|\varepsilon_{1,n}(x_i)| \leqslant (b-a)\left[\max_{0\leqslant i\leqslant n}\kappa_{1,0}(K_1,h)(x_i)\|u\|_{C^1[a,b]} + M_1\kappa_1(u,h)\right],$$

$$|\varepsilon_{2,n}(x_i)| \leqslant (b-a)\left[\max_{0\leqslant i\leqslant n}\kappa_{1,0}\Big(\partial_x K_1,h\Big)(x_i)\|u\|_{C^1[a,b]} + M_1\kappa_1(u,h)\right].$$

And finally to:

$$\max_{0\leqslant i\leqslant n}|\varepsilon_{1,n}(x_i)| + \max_{0\leqslant i\leqslant n}|\varepsilon_{2,n}(x_i)| \leqslant (b-a)\left[\max_{0\leqslant i\leqslant n}\kappa_{1,1}(K_1,h)(x_i)\|u\|_{C^1[a,b]} + 2M_1\ \kappa_1(u,h)\right].$$

As a result, we obtain

$$\|\varepsilon_n\|_{\mathbb{R}^{2n+2}} \leqslant (b-a)\left[\max_{0\leqslant i\leqslant n}\kappa_{1,1}(K_1,h)(x_i)\|u\|_{C^1[a,b]} + 2M_1\ \kappa_1(u,h)\right].$$

Theorem 7. *Let $n \geqslant 1$ and $\bar{\varepsilon}_n$ be a vector of $\mathbb{R}^{2n+2}$*

$$\bar{\varepsilon}_n = \big(\bar{\varepsilon}_{1,n}(x_0), \bar{\varepsilon}_{1,n}(x_1), \ldots, \bar{\varepsilon}_{1,n}(x_n), \bar{\varepsilon}_{2,n}(x_0) + \bar{\varepsilon}_{3,n}(x_0), \bar{\varepsilon}_{2,n}(x_1) + \bar{\varepsilon}_{3,n}(x_1), \ldots, \bar{\varepsilon}_{2,n}(x_n) + \bar{\varepsilon}_{3,n}(x_n)\big)^t, \text{ then}$$

$$\|\bar{\varepsilon}_n\|_{\mathbb{R}^{2n+2}} \leqslant 4\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\|p\|_{W^{1,1}(0,b-a)}.$$

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Proof For large n enough,

$$\left|\bar{\varepsilon}_{1,n}(x_j)\right| = \left|\int\limits_a^b p(|x_j-t|)K_2(x_j,t)u'(t)dt \;-\sum_{i=0}^n \bar{\omega}_{1,i}(x_j)K_2(x_j,x_i)u'(x_i)\right| =$$

$$= \int\limits_a^b p(|x_j-t|)\left[K_2(x_j,t)u'(t) - \frac{(x_j-t)}{h}K_2(x_j,x_{i-1})u(x_{i-1}) - \frac{(t-x_{i-1})}{h}K_2(x_j,x_i)u(x_i)\right] dt \leqslant$$

$$\leqslant \int\limits_a^b \left|p(|x_j-t|)\right| \left[\frac{|x_i-t|}{h}M_2\kappa_0(u',h) + \frac{|x_i-t|}{h}\max_{0\leqslant j\leqslant n}\kappa_{1,0}(K_2,h)(x_j)\|u\|_{C^1[a,b]} + \right.$$

$$\left. + \frac{|t-x_{i-1}|}{h}M_2\kappa_0(u',h) + \frac{|t-x_{i-1}|}{h}\max_{0\leqslant j\leqslant n}\kappa_{1,0}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right] dt.$$

Then

$$|\bar{\varepsilon}_{1,n}(x_j)| \leqslant 2\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\int\limits_a^b \left|p(|x_i-t|)\right|dt. \tag{60}$$

In same way, we get

$$|\bar{\varepsilon}_{2,n}(x_j)| \leqslant 2\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\int\limits_a^b \left|p(|x_i-t|)\right|dt, \tag{61}$$

$$|\bar{\varepsilon}_{3,n}(x_j)| \leqslant 2\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\int\limits_a^b \left|p'(|x_i-t|)\right|dt. \tag{62}$$

Summing (61) and (62), we find that

$$|\bar{\varepsilon}_{2,n}(x_j) + \bar{\varepsilon}_{3,n}(x_j)| \leqslant 2\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\|p\|_{W^{1,1}(0,b-a)}. \tag{63}$$

Finally, it follows from (60) and (63) that

$$\|\bar{\varepsilon}_n\|_{\mathbb{R}^{2n+2}} \leqslant 4\left[M_2\kappa_1(u,h) + \max_{0\leqslant j\leqslant n}\kappa_{1,1}(K_2,h)(x_j)\|u\|_{C^1[a,b]}\right]\|p\|_{W^{1,1}(0,b-a)}.$$

Theorem 8. *We have* $\lim\limits_{n\to+\infty} err_n = 0$, *where* err_n *is the local discrete error given by*

$$err_n = \max_{0\leqslant i\leqslant n}|u(x_i) - u_i| + \max_{0\leqslant i\leqslant n}|u'(x_i) - u'_i|. \tag{64}$$

Proof For large n enough, we have

$$|\lambda|\,|u(x_i) - u_i| \leqslant |\varepsilon_{1,n}(x_i)| + |\bar{\varepsilon}_{1,n}(x_i)|\big(M_1(b-a) + M_2\|p\|_{W^{1,1}(0,b-a)}\big)err_n,$$

$$|\lambda|\,|u'(x_i) - u'_i| \leqslant |\varepsilon_{1,n}(x_i)| + |\bar{\varepsilon}_{2,n}(x_i) + \bar{\varepsilon}_{3,n}(x_i)|\big(M_1(b-a) + M_2\|p\|_{W^{1,1}(0,b-a)}\big)err_n.$$

Then

$$err_n \leqslant 2\frac{M_1(b-a) + M_2\|p\|_{W^{1,1}(0,b-a)}}{|\lambda|}\left[\|\varepsilon_n\|_{\mathbb{R}^{2n+2}} + \|\bar{\varepsilon}\|_{\mathbb{R}^{2n+2}}\right],$$

and when $n\to\infty$ we get $err_n \to 0$.

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Table 2. The error between the exact and approximate solution of equation (65)

n	Cubic b-spline	Time (seconds)	Product integration	Time (seconds)
10	2.5258e-05	0.404550	1.5509e-04	0.566668
50	3.9863e-07	5.012754	6.2018e-06	10.075807
100	6.8110e-08	18.880831	1.5504e-06	37.706229
200	1.1746e-08	67.1629	3.8761e-07	134.325839

6. Numerical Test. To evaluate our numerical treatments, we consider the following example

$$\forall x \in [0,1]: \quad \lambda u(x) = \int_0^1 \frac{u(t)}{(x+t^3)^2+1}\,dt + \int_0^1 \sqrt{|x-t|}u'(t)\,dt + f(x), \tag{65}$$

where $|\lambda| = |8i+6| = 10$ and

$$f(x) = (8i+6)x^2 - \frac{1}{3}\Big(\arctan(1+x) - \arctan(x)\Big) - \frac{4}{15}\Big((2x+3)(1-x)^{\frac{3}{2}} + 2x^{\frac{5}{2}}\Big).$$

In this example $p(x,t) = \sqrt{|x-t|}$, $|K_i(x,t)| \leqslant M_i$ for $i=1,2$, where $M_1 = 2$, $M_2 = 1$. The exact solution of (65) is $u(x) = x^2$.

If $|\lambda|$ satisfies the hypothesis that $10 > 2((b-a)M_1 + M_2||p||_{W^{1,1}0,b-a})$, then the equation (65) will have a unique solution. The absolute errors $err = \max\limits_{0\leqslant i\leqslant n} |u(x_i) - u_i|$ of the proposed methods are shown in the table 2.

Below we present the figure 1, demonstrating the behavior of the absolute errors of the numerical solutions of (65) in two considered approaches.

Let us consider the next example

$$\forall x \in [0,10]: \quad \lambda u(x) = \int_0^{10} \frac{u(t)}{\sqrt{t^2+x+1}}\,dt + \int_0^{10} \sqrt{|x-t|}u'(t)\,dt + f(x) \tag{66}$$

with

$$f(x) = 60x - \sqrt{101+x} + \sqrt{1+x} - \frac{2}{3}(10-x)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}},$$

and $|\lambda| = 60$, $p(x,t) = \sqrt{|x-t|}$, $|K_i(x,t)| \leqslant M_i$ for $i=1,2$, where $M_1 = 2, M_2 = 1$. The exact solution of (66) is $u(x) = x$.

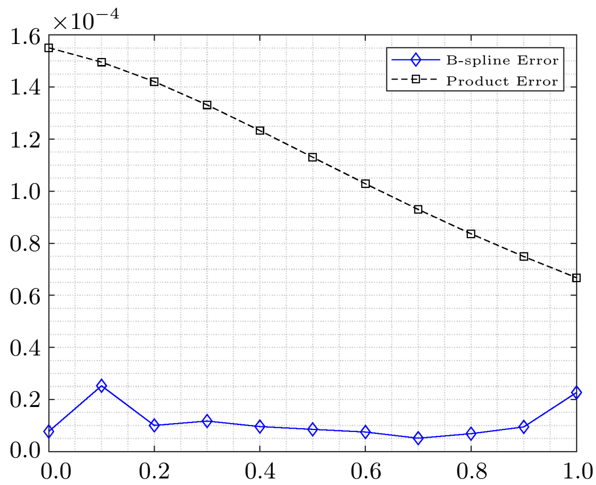

Figure 1. The absolute errors of the numerical solutions of (65), obtained within the cubic b-spline method with $n = 10$ and the product integration

Table 3. The error between the exact and approximate solution of equation (66)

n	Cubic b-spline	Time (seconds)	Product integration	Time (seconds)
50	5.3291e-15	4.867362	1.0658e-14	9.230699
100	6.2172e-15	19.957707	9.2258e-15	36.143670
200	7.1054e-15	84.333753	7.1054e-15	147.454297

If $|\lambda|$ satisfies the assumption that $60 > 2((b-a)M_1 + M_2||p||_{W^{1,1}0,b-a})$, then the equation (66) will have a unique solution. In the table 3 we present the values of the errors for various numbers n of discretization.

The corresponding absolute errors of the numerical solutions of (66) are shown in the figure 2.

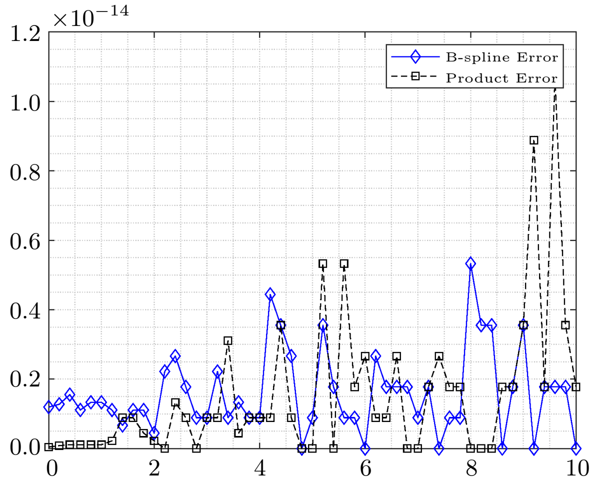

Figure 2. The absolute errors of the numerical solutions of (66), obtained within the cubic b-spline method with $n = 50$ and the product integration

Let us turn to one more example

$$\forall x \in [0,1]: \quad \lambda u(x) = \int_0^1 \frac{u(t)}{\exp(t)+x}\,dt + \int_0^1 \exp(x-t)|x-t|^{\frac{1}{3}} u'(t)\,dt + f(x), \tag{67}$$

where

$$f(x) = 5\exp(x) - \log(x+\exp(1)) + \log(1+x) - \frac{2}{4}\exp(x)x^{\frac{4}{3}} - \frac{3}{4}(1-x)^{\frac{4}{3}},$$

and $|\lambda| = 5$, $p(x,t) = |x-t|^{\frac{1}{3}}$, $|K_i(x,t)| \leqslant M_i$, for $i = 1, 2$, where $M_1 = 1$, $M_2 = 1$. The exact solution of (67) is $u(x) = \exp(x)$.

If $|\lambda|$ satisfies the hypothesis that $5 > 2((b-a)M_1 + M_2||p||_{W^{1,1}0,b-a})$, then the equation (67) will have a unique solution. In the table 4 we present the values of the errors for different numbers n of discretization.

Below we plot the figure 3, demonstrating the behavior of the absolute errors of the numerical solutions of (67) in two studied approaches.

Table 4. The error between the exact and approximate solution of equation (67)

n	Cubic b-spline	Time (seconds)	Product integration	Time (seconds)
10	9.8991e-05	1.033524	4.8240e-04	1.171200
50	1.7845e-06	5.664957	1.9252e-05	11.973596
100	3.3413e-07	27.674090	4.8123e-06	43.865372
200	6.3835e-08	94.779000	1.2030e-06	153.695566

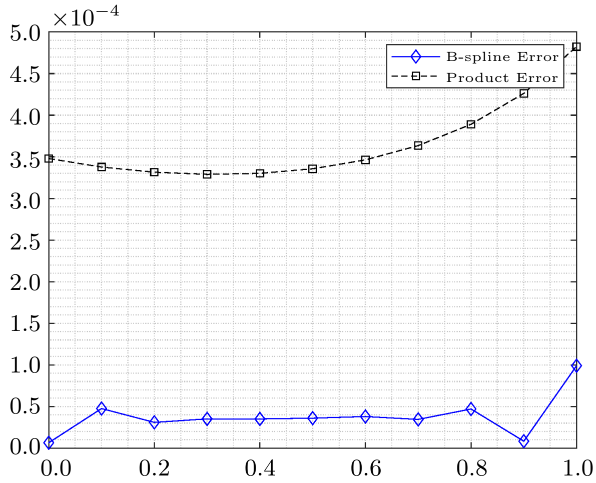

Figure 3. The absolute errors of the numerical solutions of (67), obtained within the cubic b-spline method with $n = 10$ and the product integration.

Based on the error values of the two considered approximate methods and on the used machine time, we conclude that in all three examples the b-spline collocation method is better than the product integration one. Furthermore, in contrast to the latter one, the convergence of the cubic b-spline method does not depend on the sufficient condition in theorem 5.

7. Conclusion. In this paper, we have investigated the numerical solutions of the linear integral Fredholm equation with a weakly singular kernels (1). For this goal, we have considered two approximate methods for its solution: the b-spline collocation and the product integration method. In the first of them, we have used the principle of the collocation method with b-spline bases, which led us to a conversion of our equation into the system containing $n + 1$ equations. In the second one, based on application of the product integration method, we have obtained the algebraic system containing $2n + 2$ equations. One should emphasize that the number of equations plays an important role in saving time in the resolution. This fact is a positive point for the b-spline collocation method which is presented in the numerical test. We have provided theorems showing the convergence of the b-spline collocation solution and its product integration counterpart to the exact solution. Through the numerical example we have noticed that the first method is better in terms of convergence and efficiency. We have provided our studies by the certain assumptions enabling to construct the sufficient condition which ensures the existence and uniqueness of the found solution. As perspective, we will try to modify these assumptions or to study a nonlinear integro-differential equation with both weakly singular kernels.

References

1. D. Adnan Maturi, "The Adomian Decomposition Method for Solving Heat Transfer Lighthill Singular Integral Equation Using Maple," Int. J. GEOMATE **22** (89), 16–23 (2022). https://geomatejournal.com/geomate/article/view/3140. Cited May 10, 2022.
2. I. V. Alexandrova, A. A. Ivanov, and D. V. Alexandrov, "Analytical Solution of Integro-Differential Equations Describing the Process of Intense Boiling of a Superheated Liquid," Math. Methods. Appl. Sci. **13** (2021). doi 10.1002/mma.7560.
3. P. Agarwal, U. Baltaeva, and Y. Alikulov, "Solvability of the Boundary-Value Problem for a Linear Loaded Integro-Differential Equation in an Infinite Three-Dimensional Domain," Chaos Solit. Fractals **140** (2020). doi 10.1016/j.chaos.2020.110108.
4. A. A. Minakov and C. Schick, "Integro-Differential Equation for the Non-Equilibrium Thermal Response of Glass-Forming Materials: Analytical Solutions," Symmetry. **13** (2021). doi 10.3390/sym13020256.
5. L. Zhang, L. Xu, and T. Yin, "An Accurate Hyper-Singular Boundary Integral Equation Method for Dynamic Poroelasticity in Two Dimensions," (2020). https://arxiv.org/pdf/2008.07115.pdf. Cited May 10, 2022.
6. B. Gürbüz, "A Numerical Scheme for the Solution of Neutral Integro-Differential Equations Including Variable Delay," Math. Sci. **16**, 13–21 (2022). doi 10.1007/s40096-021-00388-3.

7. H. Mesgarani and P. Parmour, “Application of Numerical Solution of Linear Fredholm Integral Equation of the First Kind for Image Restoration,” Math. Sci. (2022). doi 10.1007/s40096-022-00456-2.
8. M. Ghiat and H. Guebbai, “Analytical and Numerical Study for an Integro-Differential Nonlinear Volterra Equation with Weakly Singular Kernel,” Comput. Appl. Math. **37** (4), 4661–4974 (2018). doi 10.1007/s40314-018-0597-3.
9. M. Ghiat, H. Guebbai, M. Kurulay, and S. Segni, “On the Weakly Singular Integro-Differential Nonlinear Volterra Equation Depending in Acceleration Term,” Comput. Appl. Math. **39** (2020). doi 10.1007/s40314-020-01235-2.
10. S. Segni, M. Ghiat, and H. Guebbai, “New Approximation Method for Volterra Nonlinear Integro-Differential Equation,” Asian-Eur. J. Math. **12** (2019). doi 10.1142/S1793557119500165.
11. S. Touati, M.-Z. Aissaoui, S. Lemita, and H. Guebbai, “Investigation Approach for a Nonlinear Singular Fredholm Integro-Differential Equation,” Bol. Soc. Paran. Mat. **40** (2022). doi 10.5269/bspm.46898.
12. H. Guebbai and L. Grammont, “A New Degenerate Kernel Method for a Weakly Singular Integral Equation,” Appl. Math. Comput. **230**, 414–427 (2014). doi 10.1016/j.amc.2013.12.102.
13. S. Touati, S. Lemita, M. Ghiat, and M.-Z. Aissaoui, “Solving a Nonlinear Volterra–Fredholm Integro-Differential Equation with Weakly Singular Kernels,” Fasc. Math. **62** (1), 155–168 (2019). http://fasciculi-mathematici.put.poznan.pl/. Cited May 10, 2022.
14. A. Jafarian, R. Rezaei, and A. K. Golmankhaneh, “On Solving Fractional Higher-Order Equations via Artificial Neural Networks,” Iran. J. Sci. Technol. Trans. A: Sci. (2022). doi 10.1007/s40995-021-01254-6.
15. M. Bohner, O. Tunç, and C. Tunç, “Qualitative Analysis of Caputo Fractional Integro-Differential Equations with Constant Delays,” Comp. Appl. Math. **40** (2021). doi 10.1007/s40314-021-01595-3.
16. S. Noeiaghdam, S. Micula, and J. J. Nieto, “A Novel Technique to Control the Accuracy of a Nonlinear Fractional Order Model of COVID-19: Application of the CESTAC Method and the CADNA Library,” Mathematics **9** (2021). doi 10.3390/math9121321.
17. K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework* (Springer, New York, 2001).
18. P. Assari, “Solving Weakly Singular Integral Equations Utilizing the Meshless Local Discrete Collocation Technique,” Alex. Eng. J. **57** (4), 2497–2507 (2018). doi 10.1016/j.aej.2017.09.015.
19. A. Mennouni, “A Projection Method for Solving Cauchy Singular Integro-Differential Equations,” Appl. Math. Lett. **25** (6), 986–989 (2012). https://doi.org/10.1016/j.aml.2011.11.012. Cited May 10, 2022.
20. H. Guebbai, “Regularization and Fourier Series for Fredholm Integral Equations of the Second Kind with a Weakly Singular Kernel,” Numer. Funct. Anal. Optim. **39** (2018). doi 10.1080/01630563.2017.1364753.
21. B. Tair, H. Guebbai, S. Segni, and M. Ghiat, “Solving Linear Fredholm Integro-Differential Equation by Nystöm Method,” J. Appl. Math. Comput. Mech. **20** (3), 53–64 (2021). doi 10.17512/jamcm.2021.3.05.
22. B. Tair, H. Guebbai, S. Segni, and M. Ghiat, “An Approximation Solution of Linear Fredholm Integro-Differential Equation Using Collocation and Kantorovich Methods,” J. Appl. Math. Comput. (2021). doi 10.1007/s12190-021-01654-2.
23. M. K. Kadalbajoo and P. Arora, “B-Spline Collocation Method for the Singular-Perturbation Problem Using Artificial Viscosity,” Comput. Math. Appl. **57** (4), 650–663 (2009). doi 10.1016/j.camwa.2008.09.008.
24. R. Firouzdor, A. Heidarnejad Khoob, and Z. Mollaramezani, “Numerical Solution of Functional Integral Equations by Using B-Splines,” J. Linear Topol. Algebra **1** (1),45–53 (2012). https://oaji.net/articles/2014/1011-1405094276.pdf. Cited May 10, 2022.
25. M. Gholamian, J. Saberi-Nadjafi, and A. R. Soheili, “Cubic B-Splines Collocation Method for Solving a Partial Integro-Differential Equation with a Weakly Singular Kernel,” Comput. Methods Differ. Equ. **7** (3), 497–510 (2019). https://www.sid.ir/en/journal/ViewPaper.aspx?id=695385. Cited May 10, 2022.
26. J. Rashidinia, E. Babolian, and Z. Mahmood, “Spline Collocation for Fredholm Integral Equations,” Math. Sci. **5** (2), 147–158 (2011). https://www.sid.ir/en/journal/ViewPaper.aspx?id=250639. Cited May 10, 2022.
27. R. A. Adams and J. J. F. Fournier, *Sobolev Spaces* (Academic Press, Amsterdam, 2003).
28. A. Bashan, N. M. Yagmurlu, Y. Ucar, and A. Esen, “An Effective Approach to Numerical Solutions for the Schrödinger Equation Via Modified Cubic B-Spline Differential Quadrature Method,” Chaos Solit. Fractals **100**, 45–56 (2017). doi 10.1016/j.chaos.2017.04.038.
29. N. Ebrahimi and J. Rashidinia, “Spline Collocation for Fredholm and Volterra Integro-Differential Equations,” Int. J. Math. Model. Comput. **4** (3), 289–298 (2014). https://www.sid.ir/en/journal/ViewPaper.aspx?id=412805. Cited May 10, 2022.

30. Kh. Maleknejad and Y. Rostami, "B-Spline Method for Solving Fredholm Integral Equations of the First Kind," Int. J. Ind. Math. **11** (1), 63–70 (2019). https://www.sid.ir/en/journal/ViewPaper.aspx?id=715404. Cited May 10, 2022.
31. G. Birkhoff and C. de Boor, "Error Bounds for Spline Interpolation," J. Math. Mech. **13** (5), 827–835 (1964). http://www.jstor.org/stable/24901235. Cited May 10, 2022.
32. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, *The Theory of Splines and Their Applications* (Academic Press, New York, 1967).
33. E. V. Shikin and I. P. Alexander, *Handbook on Splines for the User* (CRC Press, Boca Raton, 1995).
34. W. J. Kammerer and G. W. Reddien, "Local Convergence of Smooth Cubic Spline Interpolates," SIAM J. Numer. Anal. **9** (4), 687–694 (1972). doi 10.1137/0709057.
35. P. M. Prenter, *Splines and Variational Methods* (Wiley, New York, 1989).
36. T. R. Lucas, "Error Bounds for Interpolating Cubic Splines under Various End Conditions," SIAM J. Numer. Anal. **11** (3), 569–584 (1974). doi 10.1137/0711049.
37. Yu. S. Volkov, "Convergence Analysis of an Interpolation Process for the Derivatives of a Complete Spline," J. Math. Sci. **187** (1), 101–114 (2012). doi 10.1007/s10958-012-1053-3.
38. G. D. Andria, G. D. Byrne, and C. A. Hall, "Convergence of Cubic Spline Interpolants of Functions Possessing Discontinuities," J. Approx. Theory **8** (2), 150–159 (1973). doi 10.1016/0021-9045(73)90024-5.
39. K. E. Atkinson, "On the Order of Convergence of Natural Cubic Spline Interpolation," SIAM J. Numer. Anal, **5** (1), 89–101 (1968). doi 10.1137/0705007.
40. Yu. S. Volkov, "Obtaining a Banded System of Equations in Complete Spline Interpolation Problem Via *B*-Spline Basis," Cent. Eur. J. Math. **10** (1), 352–356 (2012). doi 10.2478/s11533-011-0104-1.
41. L. L. Schumaker, *Spline Functions: Basic Theory* (Wiley, New York, 1981).
42. M. Ahues, A. Largillier, and B. V. Limaye, *Spectral Computations for Bounded Operators* (CRC Press, Boca Raton, 2001).
43. S. M. Zemyan, *The Classical Theory of Integral Equations* (Birkhäuser/Springer, New York, 2012).

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