doi 10.26089/NumMet.v22r105

CABARET SCHEME WITH IMPROVED DISPERSION PROPERTIES FOR SYSTEMS OF LINEAR HYPERBOLIC-TYPE DIFFERENTIAL EQUATIONS

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A conservative-characteristic method to solve systems of linear hyperbolic-type partial differential equations is proposed. This method has the fourth order of approximation on uniform grids and the second approximation order and improved dispersion properties on non-uniform grids. The proposed method is based on the well-known CABARET scheme whose conservative phases are modified by adding anti-dispersive terms of a special type. Previously, a method with similar properties was proposed only for the simplest one-dimensional linear advection equation. The modification of the scheme allows us to improve the dispersion properties of the advection for all Riemann invariants of the system of equations under consideration at once. The scheme is non-dissipative when the monotonization procedures are not used and is stable at Courant numbers CFL $\leqslant 1$. The accuracy of the method and its order of convergence are shown in a series of solving the problem of advection of a wave modulated by a Gaussian on a sequence of condensing grids. The proposed method is planned to be used as a basis for constructing a CABARET scheme with improved dispersion properties for systems of nonlinear differential equations.

Keywords: conservative-characteristic methods, CABARET scheme, Computational Fluid Dynamics (CFD), high-order schemes, hyperbolic-type equations.

1. Introduction. Numerical solution of the systems of partial differential equations of hyperbolic type is one of the main problems in mathematical modelling of physical processes. Hydrodynamics, aeroacoustics, oceanology, nuclear power and other problems of industrial mathematics are described by this type of equations. Many methods for solving equations of hyperbolic type are based on solving linear advection equations. Even in the case of such relatively simple equations, difference schemes often encounter problems with phase and amplitude conservation of propagating waves being distorted by scheme dispersion and dissipation.

Modern approaches to solve equations of hyperbolic type use a variety of different methods ranging from TVD/B (Total Variation Diminishing / Bounded) type methods that combine flux correction methods with high-order polynomial variable reconstructions for a more accurate modeling of wave propagation (e.g., WENO and the discontinuous Galerkin method [1–4]) and ending with the pseudo-spectral methods using complex computational templates to improve the dispersion and dissipation properties of the schemes [5].

One of the most dynamically developing approaches to solve hyperbolic-type systems is the use of conservative-characteristic methods that take into account the conservativity of such equations as well as the transfer of Riemann invariants over the corresponding characteristics of these systems [6]. A representative of the class of such methods is the CABARET scheme [7, 8]. This scheme has the second order of approximation in time and space and is non-dissipative (i.e., has the property of time reversibility) when the monotonic invariant procedures are excluded as well as has the minimal computational template in a single space-time cell. The latter property enables one to scale the algorithm efficiently, so the method can be run on distributed memory systems to compute a variety of large-scale resource intensive problems in hydrodynamics [9], aeroacoustics [10, 11] and dynamic elasticity [12].

The standard second-order CABARET approximation scheme is stable for Courant numbers CFL $\leqslant 1$. The monotonicity of the scheme is achieved by introducing a nonlinear flux correction procedure based on

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the maximum principle [13]. The dispersion qualities of the scheme become worse as the Courant number approaches zero. When this scheme is applied to the systems of nonlinear equations or to the systems of linear equations on non-uniform grids, the Courant number is local for each grid cell and is determined on the basis the local eigenvalues (the propagation velocities of the Riemann invariant) of the system under consideration in a given cell on a given time step. Thus, small local Courant numbers and, consequently, large dispersion errors arise in those intervals where the solution propagates most slowly. Hence, it is extremely important to improve the dispersion qualities of the CABARET scheme.

A CABARET scheme with improved dispersion properties for the simplest one-dimensional linear advection equation was proposed in [14]. The main idea of the paper was to increase the approximation of the scheme to the fourth order by introducing anti-dispersion correction in the form of a third difference derivative with some coefficient. For a long time, the results of this work were strictly theoretical because of the difficulty to generalize the proposed method to the case of a system of nonlinear differential equations. Such a generalization was proposed recently in [15] based on the addition of the third difference derivatives of the fluxes to the balance phases of the algorithm. This approach improves the accuracy of the method, but, as is shown in this paper, it improves the dispersion properties only for the fastest transferable Riemann invariant of the system under consideration.

This paper proposes a new explicit conservative-characteristic method to solve systems of linear partial differential equations of hyperbolic type; this method has the fourth order of approximation on uniform grids and the second order approximation on non-uniform grids. The method is based on adding anti-dispersive terms to the conservative phases of the standard CABARET scheme, so as to improve the dispersive transfer properties of each Riemann invariant in the system. Like the second-order CABARET scheme, the proposed method is non-dissipative (without monotonization) and is stable for Courant numbers CFL $\leqslant 1.0$. This improvement increases the method's computational template from one computational cell to three without an excessive complication of the upscaling algorithm to the case of distributed memory systems. This scheme is another step toward the improvement of the dispersion properties of the CABARET scheme and is planned to be further generalized to the more complex cases of nonlinear differential equations.

The proposed conservative-characteristic method was tested on the Gaussian-modulated wave advection problem. The numerical results were compared with those obtained by the standard CABARET scheme and the scheme proposed in [15]. The convergence of all the three methods as well as the convergence of the fourth order first method and the other second order methods are shown on a sequence of condensing uniform grids.

2. One-dimensional linear advection equation.. Let us consider the scalar linear one-dimensional advection equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{1}$$

Here x is the spatial variable, t is time, u is the advected variable, $c = \text{const}$ is the advection velocity. Equation (1) is the simplest representative of the systems of hyperbolic-type partial differential equations, i.e., the systems having a complete set of Riemann invariants. In the case of equation (1), the single Riemann invariant R coincides with the advected value: $R = u$.

2.1. A three-layer CABARET scheme and its dispersion improvement. The CABARET difference scheme was originally proposed specifically to solve equation (1) in its three-layer form [7]

$$\frac{1}{2}\left(\frac{u_{i+1}^{n+1} - u_{i+1}^{n}}{\tau} + \frac{u_{i}^{n} - u_{i}^{n-1}}{\tau}\right) + c\frac{u_{i+1}^{n} - u_{i}^{n}}{h_{i+1/2}} = 0, \tag{2}$$

where $\tau = t_{n+1} - t_n = \text{const}$ is the grid step in time and $h_{i+1/2} = x_{i+1} - x_i$ is the grid step in space. Note that in this form the CABARET scheme coincides with the Upwind LeapFrog scheme [16] produced by A. Iserles a few years before the paper [7] was published.

The scheme expressed by (2) has the second order of approximation in time and space, is stable for Courant numbers CFL $\leqslant 1$ (CFL $= c\tau/h$), is accurate on uniform grids at CFL $= 0.5$ and CFL $= 1$, and is also non-dissipative, i.e., time reversible. Using the differential approximation method [17], it can be shown that scheme (2) approximates the equation on a uniform grid:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \frac{ch^2}{12}(1 - \text{CFL})(1 - 2\text{CFL})\frac{\partial^3 u}{\partial x^3} + \ldots = 0. \tag{3}$$

Based on the differential approximation (3), we can conclude that to increase the approximation order of scheme (2) to the fourth order, the scheme's equation must be supplemented with the third difference derivative with

a certain coefficient. This improvement of the three-layer CABARET scheme on uniform grids was proposed in [14] and, then, was generalized to the case of non-uniform grids in [15]:

$$\frac{1}{2}\left(\frac{u_{i+1}^{n+1}-u_{i+1}^{n}}{\tau}+\frac{u_{i}^{n}-u_{i}^{n-1}}{\tau}\right)+c\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}} + \frac{c}{h_{i+1/2}}\left[\frac{\mu_{i+1}(c,\tau)}{\hbar_{i+1}}\left(\frac{u_{i+2}^{n}-u_{i+1}^{n}}{h_{i+3/2}}-\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}\right)-\frac{\mu_{i}(c,\tau)}{\hbar_{i}}\left(\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}-\frac{u_{i}^{n}-u_{i-1}^{n}}{h_{i-1/2}}\right)\right]=0, \tag{4}$$

where $\hbar_i = 0.5\big(h_{i+1/2}+h_{i-1/2}\big)$, $\mu_i(c,\tau) = \big([h_{i+1/2}]^2 - 3c\tau\hbar_i + 2c^2\tau^2\big)/12$. In [15] it was shown that scheme (4) has the second order of approximation together with improved dispersion properties on non-uniform grids and has the fourth order of approximation on uniform grids. Moreover, in the case of uniform grids and CFL $= 0.5$ and CFL $= 1$ it coincides with scheme (2) and, therefore, is accurate.

2.2. The conservative-characteristic form of the CABARET scheme and its dispersion improvement. When generalizing schemes (2) and (4) to the hyperbolic-type nonlinear differential equations, the above three-layer representation is converted by introducing an additional variable in the cell to a two-layer (conservative-characteristic) one consisting of three phases [18]. For scheme (2), this representation takes the form

$$\frac{u_{i+1/2}^{n+1/2}-u_{i+1/2}^{n}}{\tau_n/2}+c\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}=0, \tag{5}$$

$$\begin{aligned} u_i^{n+1} &= 2u_{i-1/2}^{n+1/2}-u_{i-1}^{n}, \quad \text{if } c>0,\\ u_i^{n+1} &= 2u_{i+1/2}^{n+1/2}-u_{i+1}^{n}, \quad \text{if } c<0, \end{aligned} \tag{6}$$

$$\frac{u_{i+1/2}^{n+1}-u_{i+1/2}^{n+1/2}}{\tau_n/2}+c\frac{u_{i+1}^{n+1}-u_{i}^{n+1}}{h_{i+1/2}}=0. \tag{7}$$

Here $\tau_n = t_{n+1} - t_n$ are the grid steps in time, $u_i^\bullet$ are the "flux" values of u given at the grid nodes, $u_{i+1/2}^\bullet$ are the "conservative" values of u given at the grid cell centers. The first (5) and the third (7) phases are called the conservative phases and represent the grid conservation laws of the value u. The second phase (6) is the characteristic phase extrapolating the Riemann invariant value $R = u$ in the direction of its corresponding characteristic $x - ct = \text{const}$. In the case of a constant time step, the scheme expressed by (5)–(7) is reduced to scheme (2) by eliminating the conservative variables $u_{i+1/2}^\bullet$.

The scheme (5)–(7), like its three-layer analog (2), has the second-order of approximation in time and space as well as the time reversibility property. We should note the minimality of its template: only one space-time cell. When scaling the scheme for parallel computing systems with distributed memory, the neighboring partitions only need to exchange the values of Riemann invariants arriving at their boundaries at each time step, which greatly simplifies the scaling of algorithms.

The conservative-characteristic representation of (4) can be obtained in a similar way:

$$\frac{u_{i+1/2}^{n+1/2}-u_{i+1/2}^{n}}{\tau_n/2}+c\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}+c\langle\mu(c,\tau_n)\frac{\partial^3 u}{\partial x^3}\rangle_{i+1/2}^{n}=0, \tag{8}$$

$$\begin{aligned} u_i^{n+1} &= 2u_{i-1/2}^{n+1/2}-u_{i-1}^{n}, \quad \text{if } c>0,\\ u_i^{n+1} &= 2u_{i+1/2}^{n+1/2}-u_{i+1}^{n}, \quad \text{if } c<0, \end{aligned} \tag{9}$$

$$\frac{u_{i+1/2}^{n+1}-u_{i+1/2}^{n+1/2}}{\tau_n/2}+c\frac{u_{i+1}^{n+1}-u_{i}^{n+1}}{h_{i+1/2}}+c\langle\mu(c,\tau_n)\frac{\partial^3 u}{\partial x^3}\rangle_{i+1/2}^{n+1}=0, \tag{10}$$

$$\langle\mu(c,\tau)\frac{\partial^3 u}{\partial x^3}\rangle_{i+1/2}^{n} = \frac{1}{h_{i+1/2}}\left[\frac{\mu_{i+1}(c,\tau)}{\hbar_{i+1}}\left(\frac{u_{i+2}^{n}-u_{i+1}^{n}}{h_{i+3/2}}-\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}\right) - \frac{\mu_{i}(c,\tau)}{\hbar_{i}}\left(\frac{u_{i+1}^{n}-u_{i}^{n}}{h_{i+1/2}}-\frac{u_{i}^{n}-u_{i-1}^{n}}{h_{i-1/2}}\right)\right]. \tag{11}$$

The scheme expressed by (8)–(11) can also be reduced to scheme (4) by eliminating the conservative variables $u_{i+1/2}^\bullet$; this system inherits all the properties of the original scheme. The formulas above show that

in order to improve the dispersion properties of the scheme expressed by (5)–(7), we should modify the first phase (8) and the third phase (10) of the scheme only. In this case, when the algorithm is scaled for the computer systems with distributed memory, the neighboring partitions have to exchange not only the values of the Riemann invariants but also the values of flux variables u_i^{n+1} (one layer of flux variables on each side).

3. Systems of linear hyperbolic-type one-dimensional differential equations. Let us consider the system of one-dimensional linear partial differential equations of the first order solved with respect to time derivatives:

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0, \quad \mathbf{u} = \{u^{(j)}\} \in \mathbb{R}^k, \quad A \in \mathbb{R}^{k\times k}, \quad A = \text{const}. \tag{12}$$

Let us assume that the system expressed by (12) is hyperbolic, i.e., the matrix A has a complete set of real eigenvalues $\{\lambda^{(j)}\} \in \mathbb{R}$ and a complete set of eigenvectors. Then, this system can be decomposed into k independent advection equations:

$$\frac{\partial R^{(j)}}{\partial t} + \lambda^{(j)}\frac{\partial R^{(j)}}{\partial x} = 0, \quad j = \overline{1,k}, \tag{13}$$

where $\mathbf{R} = \{R^{(j)}\} \in \mathbb{R}^k$, $R^{(j)} = (\mathbf{l}^{(j)}, \mathbf{u})$ are the Riemann invariants, $\mathbf{l}^{(j)} \in \mathbb{R}^k$ are the left eigenvector rows of the matrix A corresponding to the eigenvalues $\lambda^{(j)}$.

3.1. The standard CABARET scheme for systems of equations. The CABARET conservative-characteristic scheme for system (12) has the following form [19]:

$$\frac{\mathbf{u}_{i+1/2}^{n+1/2} - \mathbf{u}_{i+1/2}^{n}}{\tau_n/2} + A\frac{\mathbf{u}_{i+1}^{n} - \mathbf{u}_{i}^{n}}{h_{i+1/2}} = 0, \tag{14}$$

$$\begin{aligned} &[R^{(j)}]_i^{n+1} = 2[R^{(j)}]_{i-1/2}^{n+1/2} - [R^{(j)}]_{i-1}^{n}, \quad \text{if } \lambda^{(j)} > 0, \\ &[R^{(j)}]_i^{n+1} = 2[R^{(j)}]_{i+1/2}^{n+1/2} - [R^{(j)}]_{i+1}^{n}, \quad \text{if } \lambda^{(j)} < 0, \end{aligned} \qquad j = \overline{1,k}, \tag{15}$$

$$\frac{\mathbf{u}_{i+1/2}^{n+1} - \mathbf{u}_{i+1/2}^{n+1/2}}{\tau_n/2} + A\frac{\mathbf{u}_{i+1}^{n+1} - \mathbf{u}_{i}^{n+1}}{h_{i+1/2}} = 0. \tag{16}$$

Here the transition to the Riemann invariants $\mathbf{R}_\star^\bullet = L\mathbf{u}_\star^\bullet$ is performed when the first phase (14) is completed, whereas the return to the original variables is performed after the second phase (15): $\mathbf{u}_\star^\bullet = L^{-1}\mathbf{R}_\star^\bullet$, where $L \in \mathbb{R}^{k\times k}$ is a matrix of the left eigenvectors $\mathbf{l}^{(j)}$. In this case, the Courant number is defined as follows: $\text{CFL} = \max\left(|\lambda^{(j)}| \cdot \tau_n/h_{i+1/2}\right)$.

The scheme expressed by (14)–(16) has the same properties as the scheme expressed by (5)–(7) for the advection equation: the second order of approximation and the time reversibility. Indeed, multiplying the equations of the first (14) and third (16) phases from the left by the eigenvector $\mathbf{l}^{(j)}$, we obtain equations (5), (7) of the scheme for the advection equation for $R^{(j)}$ with the velocity $\lambda^{(j)}$. Thus, due to the linear dependence of Riemann invariants on the original variables of the problem, scheme (14)–(16) is equivalent to k independent schemes (5)–(7) for Riemann invariants $R^{(j)}$.

3.2. CABARET scheme with improved dispersion properties for systems of equations. In order to improve the dispersion properties of scheme (14)–(16) for system (12), we use the same idea as for scheme (8)–(11): we add anti-dispersion terms to the conservative phases of the algorithm without changing the characteristic phase. To accomplish this, we consider the conservative phases of scheme (8) with improved dispersion properties for each of the Riemann invariant advection equations (13):

$$\frac{[R^{(j)}]_{i+1/2}^{n+1/2} - [R^{(j)}]_{i+1/2}^{n}}{\tau_n/2} + \lambda^{(j)}\frac{[R^{(j)}]_{i+1}^{n} - [R^{(j)}]u_i^n}{h_{i+1/2}} + \lambda^{(j)}\left\langle \mu(\lambda^{(j)}, \tau_n)\frac{\partial^3 R^{(j)}}{\partial x^3}\right\rangle_{i+1/2}^{n} = 0, \quad j = \overline{1,k}. \tag{17}$$

Then, we return to the original variables $\mathbf{u}$ of the system by multiplying the vector equation (17) from the left by the matrix L^{-1}:

$$\frac{\mathbf{u}_{i+1/2}^{n+1/2} - \mathbf{u}_{i+1/2}^{n}}{\tau_n/2} + A\frac{\mathbf{u}_{i+1}^{n} - \mathbf{u}_{i}^{n}}{h_{i+1/2}} + \left\langle L^{-1}\Lambda D L\frac{\partial^3 \mathbf{u}}{\partial x^3}\right\rangle_{i+1/2}^{n} = 0.$$

Here $\Lambda \in \mathbb{R}^{k\times k}$ is the diagonal matrix formed by the eigenvalues of the system $(\Lambda)_{ij} = \delta_{ij}\lambda^{(j)}$ and $D \in \mathbb{R}^{k\times k}$ is the diagonal matrix of anti-dispersion coefficients $(D)_{ij} = \delta_{ij}\mu(\lambda^{(j)}, \tau_n)$. The equations with the anti-dispersion

correction terms for the third phase (16) are obtained in a similarly way. The CABARET scheme with improved dispersion properties for systems of linear equations of hyperbolic type has the following form:

$$\frac{\mathbf{u}_{i+1/2}^{n+1/2}-\mathbf{u}_{i+1/2}^{n}}{\tau_n/2}+A\frac{\mathbf{u}_{i+1}^{n}-\mathbf{u}_{i}^{n}}{h_{i+1/2}}+\left\langle L^{-1}\Lambda DL\frac{\partial^3\mathbf{u}}{\partial x^3}\right\rangle_{i+1/2}^{n}=0, \tag{18}$$

$$\begin{aligned}&[R^{(j)}]_i^{n+1}=2[R^{(j)}]_{i-1/2}^{n+1/2}-[R^{(j)}]_{i-1}^{n}, \quad \text{if } \lambda^{(j)}>0,\\ &[R^{(j)}]_i^{n+1}=2[R^{(j)}]_{i+1/2}^{n+1/2}-[R^{(j)}]_{i+1}^{n}, \quad \text{if } \lambda^{(j)}<0,\end{aligned} \qquad j=\overline{1,k}, \tag{19}$$

$$\frac{\mathbf{u}_{i+1/2}^{n+1}-\mathbf{u}_{i+1/2}^{n+1/2}}{\tau_n/2}+A\frac{\mathbf{u}_{i+1}^{n+1}-\mathbf{u}_{i}^{n+1}}{h_{i+1/2}}+\left\langle L^{-1}\Lambda DL\frac{\partial^3\mathbf{u}}{\partial x^3}\right\rangle_{i+1/2}^{n+1}=0. \tag{20}$$

Thus, the approximation of each equation in (12) includes not only the difference analog of the first spatial derivative of the fluxes $(A\mathbf{u})^{(j)}$, but also the analog of the third difference derivative of, generally speaking, all the variables $\mathbf{u}$ of the system. Thus, for example, the third difference spatial derivative of density is included into the approximation of the mass conservation law for linearized gas dynamics equations.

The scheme expressed by (18)–(20) has the same properties as the scheme (8)–(10) for the advection equation: it has the second order of approximation as well as the improved dispersion properties on non-uniform grids and the fourth order of approximation on uniform grids. This scheme is stable for Courant numbers CFL $\leqslant 1$ and has the property of time reversibility. Indeed, multiplying the equations of the first (18) and third (20) phases from the left by the eigenvector matrix L, we come to the independent equations (8), (10) of the scheme for the advection equation for $R^{(j)}$ with velocity $\lambda^{(j)}$. Thus, due to the linear dependence of the Riemann invariants on the original variables of the problem, scheme (18)–(20) is equivalent to k independent schemes (8)–(10) for the Riemann invariants $R^{(j)}$. In this case, the system, generally speaking, is not accurate on uniform grids at Courant numbers CFL = 0.5 and CFL = 1. This is because of the fact that the Courant number for the system of equations is determined by the maximum eigenvalue in absolute value and, for example, only the most rapidly propagating invariants are transferred accurately for CFL = 1. Note also that in order to monotonize the scheme, the characteristic phase of algorithm (19) must be supplemented with the procedure for the nonlinear correction of Riemann invariants based on the maximum principle. Since the characteristic phase remains the same, the correction of the invariants should be done in the same way as in the standard CABARET scheme [19].

The proposed scheme (18)–(20) differs from that proposed in [15] in that the anti-dispersion correction is introduced for each Riemann invariant of the system separately, whereas in [15] the same anti-dispersion correction corresponding to the maximum eigenvalue in absolute value is applied to all the Riemann invariants, which makes it impossible to obtain the fourth order of approximation on uniform computational grids. Note that the formulas for the scheme [15] can be derived from formulas (18)-(20) by replacing the diagonal matrix of anti-dispersion coefficients D by the diagonal matrix with the same elements: $(\widehat{D})_{ij}=\delta_{ij}\mu(\lambda_{max},\tau_n)$, where λ_{max} is the maximum eigenvalue in absolute value in the system (12).

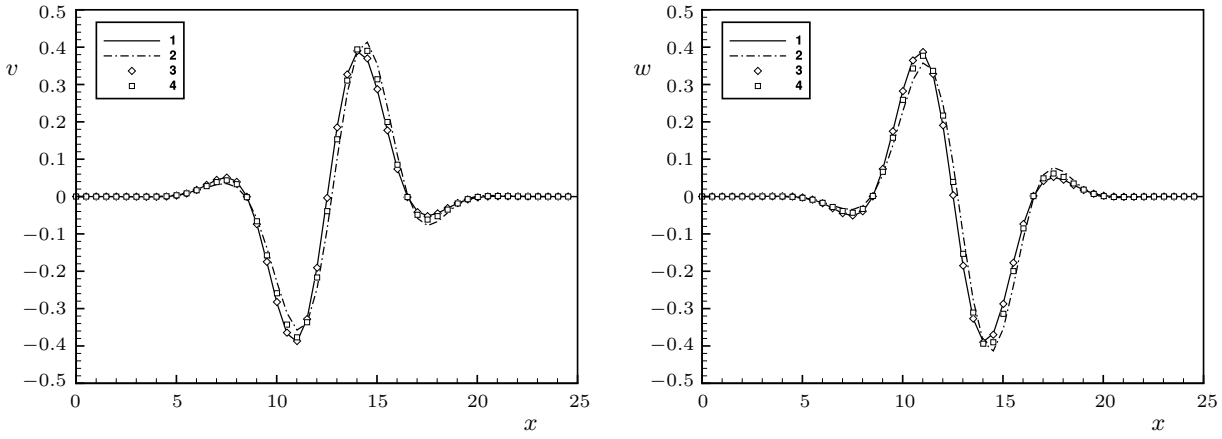
4. Test results.. In order to test the proposed scheme (18)–(20) and to show its fourth order of convergence, we consider the following system of hyperbolic-type partial differential equations:

$$\begin{aligned}&\frac{\partial v}{\partial t}+2\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}=0,\\ &\frac{\partial w}{\partial t}+2\frac{\partial w}{\partial x}+\frac{\partial v}{\partial x}=0,\end{aligned} \qquad x\in(-200,200), \quad t>0. \tag{21}$$

System (21) can be treated as an oscillation equation with the coefficient $c^2=4$ in a moving coordinate system written in the form of first-order partial derivative equations. We supplement this system with periodic boundary conditions as well as by initial conditions in the form of a Gaussian-modulated wave similar to those used earlier when studying the properties of the CABARET scheme [20, 21]:

$$\begin{aligned}&v(x,t=0)=2\sin(\pi x/4)\exp(-\ln(2)\,(x/3)^2),\\ &w(x,t=0)=\sin(\pi x/4)\exp(-\ln(2)\,(x/3)^2),\end{aligned} \qquad x\in(-200,200). \tag{22}$$

In the figure we illustrate the numerical results obtained for the problem (21), (22) at the time instant $t=12.5$ on the interval $x\in[0,25]$ using the standard CABARET scheme (14)–(16), the CABARET



Numerical results for the problem (21), (22) at the time instant $t = 12.5$ on the interval $x \in [0, 25]$ using a uniform grid of 800 cells for CFL $= 0.2$. *a)* the solution component $v(x, t)$; *b)* the solution component $w(x, t)$. 1 — analytical solution; 2 — solution by standard CABARET scheme; 3 — solution by CABARET scheme with improved dispersion properties; 4 — solution by scheme [15]

scheme (18)–(20) with improved dispersion properties, the scheme proposed in [15] as well as the analytical solution. Computations were performed on a uniform grid of 800 computational cells at the Courant number CFL $= 0.2$; the monotonization procedures were not used due to the smoothness of the solution in question. The time moment $t = 12.5$ is chosen so that the components of the solution corresponding to the Riemann invariants moving with the velocities $\lambda_1 = 3$ and $\lambda_2 = 1$ have time to move apart. Part of the solution corresponding to the slowly propagating Riemann invariant with the eigenvalue λ_2 lies on the interval $x \in [0, 25]$; in scheme [15], an anti-dispersion correction inconsistent with λ_2 is applied to this invariant.

The results shown in the figure allow us to conclude that the proposed scheme with improved dispersion properties gives a sufficiently accurate solution to the problem that almost matches the analytical solution. Although the scheme from [15] gives a more accurate result than the standard CABARET scheme, it is inferior in accuracy to the proposed method.

Tables 1–3 show the errors and orders of accuracy (OOA) for computations on different uniform grids and with different Courant numbers CFL for the standard CABARET scheme, the scheme from [15] and the CABARET scheme with improved dispersion properties, respectively. The error was calculated for the solution at the time instant $t = 12.5$ using the standard C-norm; the maximum value of the errors for the variables v and w was inserted into each table. The order of convergence was calculated using Runge's rule [22].

The results shown in Tables 1–3 allow us to conclude that all the three methods are convergent on condensing grids. The standard CABARET method has the second order of convergence; the CABARET method with improved dispersion properties has the fourth order of convergence. Although the method of [15] gives a more accurate solution than the standard CABARET scheme, it also has only the second order of convergence,

Table 1

C-norm errors and orders of accuracy (OOA) for computing the problem (21), (22) using the standard CABARET scheme

Cells	C-error CFL $= 0.2$	OOA	C-error CFL $= 0.4$	OOA	C-error CFL $= 0.6$	OOA	C-error CF $L = 0.8$	OOA
3200	2.98×10^{-2}	2.00	7.45×10^{-3}	2.00	4.96×10^{-3}	2.00	7.46×10^{-3}	2.00
1600	1.18×10^{-1}	1.99	2.98×10^{-2}	2.00	1.99×10^{-2}	2.00	2.98×10^{-2}	1.99
800	4.46×10^{-1}	1.90	1.15×10^{-1}	1.95	7.90×10^{-2}	2.00	1.20×10^{-1}	2.00
400	1.03	1.23	4.45×10^{-1}	1.90	2.61×10^{-1}	1.70	4.49×10^{-1}	1.96
200	1.00	1.01	9.48×10^{-1}	1.22	8.49×10^{-1}	1.53	9.77×10^{-1}	1.27

Table 2

C-norm errors and orders of accuracy (OOA) for computing the problem (21), (22) using the scheme [15]

Cells	C-error CFL = 0.2	OOA	C-error CFL = 0.4	OOA	C-error CFL = 0.6	OOA	C-error CFL = 0.8	OOA
3200	2.28×10^{-3}	2.00	3.56×10^{-3}	2.00	3.87×10^{-3}	2.00	3.19×10^{-3}	2.00
1600	9.28×10^{-3}	2.00	1.43×10^{-2}	2.00	1.55×10^{-2}	2.00	1.27×10^{-2}	1.99
800	3.91×10^{-2}	2.08	5.69×10^{-2}	1.99	6.06×10^{-2}	1.97	4.98×10^{-2}	1.96
400	2.66×10^{-1}	2.76	2.14×10^{-1}	1.88	2.17×10^{-1}	1.80	1.86×10^{-1}	1.83
200	8.70×10^{-1}	1.76	5.43×10^{-1}	1.47	4.58×10^{-1}	0.95	5.53×10^{-1}	1.60

Table 3

C-norm errors and orders of accuracy (OOA) for calculations of problem (21), (22) by the CABARET scheme with improved dispersion properties

Cells	C-error CFL = 0.2	OOA	C-error CFL = 0.4	OOA	C-error CFL = 0.6	OOA	C-error CFL = 0.8	OOA
3200	7.87×10^{-5}	4.00	2.04×10^{-5}	4.00	1.36×10^{-5}	4.00	1.99×10^{-5}	4.00
1600	1.26×10^{-3}	3.97	3.25×10^{-4}	3.99	2.16×10^{-4}	3.99	3.18×10^{-4}	3.99
800	2.00×10^{-2}	3.96	5.00×10^{-3}	3.97	3.40×10^{-3}	3.97	5.07×10^{-3}	3.99
400	2.66×10^{-1}	3.74	7.07×10^{-2}	3.79	4.46×10^{-2}	3.69	6.40×10^{-2}	3.69
200	8.98×10^{-1}	1.72	5.41×10^{-1}	2.94	4.52×10^{-1}	3.14	5.55×10^{-1}	2.89

which can be explained by an incomplete anti-dispersion correction for all Riemann invariants except for the fastest transferable one.

5. Conclusion. This paper presents a new explicit conservative-characteristic method of the fourth-order approximation to solve systems of linear partial differential equations of hyperbolic type. This method is based on adding special anti-dispersion terms to the standard CABARET scheme; these terms improve the dispersion transfer properties for all Riemann invariants of the system. The proposed scheme is non-dissipative when the monotonization procedures are not used and is stable when the Courant numbers CFL $\leqslant 1$. The method's accuracy and its order of convergence are demonstrated on a series of calculations for the problem of Gaussian-modulated wave transfer on a sequence of condensing grids.

The proposed scheme for linear differential equations is not only of theoretical but also of practical interest and can be used, for example, to compute problems related to detecting the thermoacoustic instability (vibrational combustion) of acoustic paths [23].

This work was supported by the Russian Foundation for Basic Research (grant 19–01–00472) and was performed at Lomonosov Moscow State University.

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Accepted
20 December 2020